Applied Mathematical Sciences, Vol. 5, 2011, no. 24, 1177 - 1191

Blow-up and Quenching for Coupled Semilinear Parabolic Systems

H. T. Liu[1]

Department of Applied Mathematics
Tatung University, 40 Chung Shan North Road
Sec. 3, Taipei, Taiwan

W. Y. Chan[2]

Department of Mathematics
Southeast Missouri State University
Cape Girardeau, MO 63701, USA

Abstract

Let $T \leq \infty$, D be a bounded n-dimensional domain, ∂D be the boundary of D. In this paper, we study the blow-up and quenching of the solutions of the following coupled semilinear parabolic systems:

$$u_t - \Delta u = \gamma v^p \text{ in } D \times (0,T),$$

$$v_t - \Delta v = \mu \frac{1}{(1-u)^q} \text{ in } D \times (0,T),$$

$$u(x,0) = u_0(x) \text{ and } v(x,0) = v_0(x) \text{ on } \bar{D},\ u(x,t) = 0 = v(x,t) \text{ on } \partial D \times (0,T),$$

where γ, μ, p, and q are positive real numbers with $p \geq 1$ and $q > 1$, and $u_0(x)$ and $v_0(x)$ are given functions. We prove that v blows up and u quenches simultaneously in a finite time under some sufficient conditions. We also discuss the blow-up and quenching rates of the solutions.

Mathematics Subject Classification: 35K55, 35K57, 35K60, 35K65

Keywords: Semilinear parabolic systems, existence of the solution, blow-up and quenching rates

---

[1]The research work of this author was partially supported by the Tatung University Research Grant under the contract B99-A02-049.

[2]The research work of this author is supported by the Grants and Research Funding of Southeast Missouri State University under the funding number 102340.

1 Introduction

Let γ and μ be positive real numbers, p and q be positive real numbers with $p \geq 1$ and $q > 1$, $T \leq \infty$, D be a bounded n-dimensional domain, ∂D and $\bar{D}$ be the boundary and closure of D respectively, $\Omega = D \times (0, T)$, $\bar{\Omega} = \bar{D} \times [0, T)$, Δ be the Laplace operator, and L be the parabolic operator such that $Lu = u_t - \Delta u$. In this paper, we study the blow-up and quenching of the following coupled semilinear parabolic systems:

$$\left.\begin{aligned} Lu &= \gamma v^p \text{ in } \Omega, \\ Lv &= \mu \frac{1}{(1-u)^q} \text{ in } \Omega, \end{aligned}\right\} \tag{1.1}$$

$$u(x,0) = u_0(x) \text{ and } v(x,0) = v_0(x) \text{ on } \bar{D},\ u(x,t) = 0 = v(x,t) \text{ on } \partial D \times (0,T), \tag{1.2}$$

where u_0 and v_0 belong to $C^{2+\alpha}\left(\bar{D}\right)$ for some $\alpha \in (0,1)$, and they are non-negative functions on $\bar{D}$ such that $u_0(x) < 1$ on $\bar{D}$ and $u_0(x) = 0 = v_0(x)$ on ∂D.

We say that x is a blow-up point and T is the blow-up time of v if there exists a sequence (x_n, t_n) such that $v(x_n, t_n) \to \infty$ when $x_n \to x$ and $t_n \to T$. The solution u is said to quench at time T if

$$\max\left\{u(x,t) : x \in \bar{D}\right\} \to 1^- \text{ when } t \to T^-.$$

Equations (1.1) arise in the chemical reaction process. It can describe the thermal ignition of chemicals when u and v represent the density of the chemical and its temperature respectively (cf. Pao [5, p. 687] and Wang [8]).

When the forcing functions γv^p and $\mu/(1-u)^q$ of (1.1) are replaced by $u^{p_1} v^{q_1}$ and $u^{p_2} v^{q_2}$, Wang [8] studied the blow-up rate of the problem (1.1)-(1.2). Guo, Sasayama, and Wang [4] obtained the blow-up rates estimates for the Cauchy problem of (1.1) when u and v blow up simultaneously or non-simultaneously. When the source term of the second equation of (1.1) equals u^q, Samarkii, Galaktionov, Kurdymov, and Mikhailov [7] obtained the results of the blow-up and global existence of the solutions of the following quasilinear systems:

$$\begin{aligned} u_t - \Delta u^{a+1} &= v^p \text{ in } \Omega, \\ v_t - \Delta v^{b+1} &= u^q \text{ in } \Omega, \end{aligned}$$

for some positive real numbers a and b.

In Section 2, we shall study the blow-up and quenching of the solutions of the problem (1.1)-(1.2). We shall give some sufficient conditions that v blows up and u quenches simultaneously. In Section 3, the blow-up rate of v and quenching rate of u shall be discussed.

2 Existence and Nonexistence of the Solution

To establish existence of the solutions of the problem (1.1)-(1.2), we assume that u_0 and v_0 satisfy

$$\left.\begin{array}{r} -\Delta u_0 \leq \gamma v_0^p \text{ in } \bar{D}, \\ -\Delta v_0 \leq \mu \dfrac{1}{(1-u_0)^q} \text{ in } \bar{D}. \end{array}\right\} \tag{2.1}$$

According to Pao [5, pp. 383-384], u_0 and v_0 are lower solutions of the problem (1.1)-(1.2). To construct upper solutions, let $F(t)$ and $I(t)$ satisfy

$$\begin{aligned} \frac{d}{dt}F(t) &= \gamma I^p(t) \text{ for } t>0 \text{ and } F(0)=m_1, \\ \frac{d}{dt}I(t) &= \mu \frac{1}{(1-F(t))^q} \text{ for } t>0 \text{ and } I(0)=m_2, \end{aligned}$$

where $m_1 = \max_{x\in\bar{D}} u_0(x)$ and $m_2 = \max_{x\in\bar{D}} v_0(x)$. There exists some $T>0$ such that the above problems have unique solutions for $t \in (0,T)$, and their solutions are given by

$$\begin{aligned} F(t) &= F(0) + \gamma \int_0^t I^p(s)ds, \\ I(t) &= I(0) + \mu \int_0^t \frac{1}{(1-F(s))^q} ds. \end{aligned}$$

Since m_1 and m_2 are nonnegative, $\mu/(1-F(t))^q$ is positive and $\gamma I^p(t)$ is nonnegative respectively. This implies that $F'(t) \geq 0$ and $I'(t) > 0$. Substitute the above expressions into the equation (1.1), we get

$$\begin{aligned} \frac{d}{dt}F - \Delta F - \gamma I^p &= \frac{d}{dt}F - \gamma I^p = 0, \\ \frac{d}{dt}I - \Delta I - \mu\frac{1}{(1-F)^q} &= \frac{d}{dt}I - \mu\frac{1}{(1-F)^q} = 0. \end{aligned}$$

In addition, $F(0) \geq u_0(x)$ and $I(0) \geq v_0(x)$ on $\bar{D}$. Thus, $F(t)$ and $I(t)$ are upper solutions of the problem (1.1)-(1.2).

By Theorem 8.3.1 of Pao [5, p. 393], the problem (1.1)-(1.2) has a unique solution (u,v) such that $u_0 \leq u \leq F$ and $v_0 \leq v \leq I$. Furthermore, by the mean value theorem, there exist ξ_1 (between v_1 and v_2) and ξ_2 (between u_1 and u_2) such that

$$|\gamma v_1^p - \gamma v_2^p| = \gamma p |\xi_1|^{p-1} |v_1 - v_2|,$$

$$\left|\frac{\mu}{(1-u_1)^q} - \frac{\mu}{(1-u_2)^q}\right| = \mu q \frac{1}{|1-\xi_2|^{q+1}} |u_1 - u_2|.$$

It follows from the Theorem 8.9.2 of Pao [5, p. 436] that u and v either exist globally or only exist in a finite time.

Lemma 2.1 *Assume that u_0 and v_0 satisfy the equation (2.1), and (u_0, v_0) is not a stationary solution of the problem (1.1)-(1.2). Then u_t and v_t are positive in Ω.*

Proof. From the condition (2.1), we know that $u(x,t) \geq u_0(x)$ and $v(x,t) \geq v_0(x)$ on $\bar{\Omega}$. For some $h \in (0,T)$, let $u_1(x,t) = u(x,t+h)$, and $v_1(x,t) = v(x,t+h)$ for $t \in (0, T-h)$. Then $u_1(x,0) \geq u(x,0)$, $v_1(x,0) \geq v(x,0)$ on $\bar{D}$, and $u_1(x,t) = 0 = v_1(x,t)$ on $\partial D \times (0, T-h)$. It follows from a direct computation that u_1 and v_1 satisfy the equations (1.1). Hence by the maximum principle that $u_1(x,t) \geq u(x,t)$ and $v_1(x,t) \geq v(x,t)$ on $\bar{D} \times [0, T-h)$. This gives

$$\frac{u(x,t+h) - u(x,t)}{h} \geq 0 \text{ and } \frac{v(x,t+h) - v(x,t)}{h} \geq 0 \text{ on } \bar{D} \times [0, T-h)\,.$$

As $h \to 0$, we get u_t and v_t are nonnegative on $\bar{\Omega}$.

To show that u_t and v_t are positive, we differentiate the equation (1.1) with respect to t and get

$$\begin{aligned} u_{tt} - \Delta u_t &= \gamma p v^{p-1} v_t \text{ in } \Omega, \\ v_{tt} - \Delta v_t &= \mu q \frac{1}{(1-u)^{q+1}} u_t \text{ in } \Omega. \end{aligned}$$

For $(x,t) \in \bar{\Omega}$, the integral representation for the solutions u_t and v_t is given by,

$$u_t(x,t) = \int_D G(x,t;\xi,0)\, u_t(\xi,0)\, d\xi + \int_0^t \int_D G(x,t;\xi,\tau)\, \gamma p v^{p-1}(\xi,\tau)\, v_\tau(\xi,\tau)\, d\xi d\tau,$$

$$\begin{aligned} v_t(x,t) &= \int_D G(x,t;\xi,0)\, v_t(\xi,0)\, d\xi \\ &\quad + \int_0^t \int_D G(x,t;\xi,\tau)\, \mu q \frac{1}{(1-u(\xi,\tau))^{q+1}} u_\tau(\xi,\tau)\, d\xi d\tau, \end{aligned}$$

where $G(x,t;\xi,\tau)$ is the Green's function of the operator L. Since $G(x,t;\xi,\tau) > 0$ in the set

$$\{(x,t;\xi,\tau) : x \text{ and } \xi \text{ are in } D, \text{ and } t > \tau \geq 0\},$$

u_t and v_t are nonnegative on $\bar{\Omega}$, and (u_0, v_0) is not a stationary solution of the problem (1.1)-(1.2), this shows that u_t and v_t are positive in Ω. □

From Theorem 3.3 of Friedman and McLeod [3] and Theorem 2.2 Deng and Levine [1], the quenching and blow-up sets of u and v respectively are compact subsets of D.

Lemma 2.2 *u_t and v_t tend to infinity somewhere inside Ω when $v \to \infty$ and $u \to 1^-$ respectively.*

Proof. Let D' be a proper subset of D which contains the quenching and blow-up points of u and v as interior points, $\overline{D'}$ be the closure of D', $\Omega' = D' \times [0, T)$, $\partial\Omega'$ be the boundary of Ω', $\bar{\Omega}'$ be the closure of Ω', κ be a small positive number, $J = u_t - \kappa\gamma v^p$, and $H = v_t - \kappa\mu/(1-u)^q$. Since $D' \subset D$, we have $u < 1$ and $v < \infty$ in $\Omega\backslash\Omega'$. Because u_t and v_t are positive in Ω', for some fixed $s \in (0, T)$, we choose κ such that J and H are nonnegative on $\partial\Omega'$ for $t \in (s, T)$, and $J(x, s)$ and $H(x, s)$ are nonnegative for $x \in \overline{D'}$. Then,

$$J_t = u_{tt} - \kappa\gamma p v^{p-1} v_t,$$

$$\Delta J = \Delta u_t - \kappa\gamma p v^{p-1}\Delta v - \kappa\gamma p (p-1) v^{p-2} (\bigtriangledown v)^2 .$$

By a direct computation,

$$\begin{aligned}
J_t - \Delta J &= u_{tt} - \Delta u_t - \kappa\gamma p v^{p-1}(v_t - \Delta v) + \kappa\gamma p (p-1) v^{p-2} (\bigtriangledown v)^2 \\
&= \gamma p v^{p-1} v_t - \kappa\gamma p v^{p-1}\frac{\mu}{(1-u)^q} + \kappa\gamma p (p-1) v^{p-2} (\bigtriangledown v)^2 \\
&= \gamma p v^{p-1} H + \kappa\gamma p (p-1) v^{p-2} (\bigtriangledown v)^2 \\
&\geq \gamma p v^{p-1} H.
\end{aligned}$$

Similarly,

$$H_t - \Delta H = \mu q (1-u)^{-(q+1)} J + \kappa\mu q (q+1)(1-u)^{-(q+2)} (\bigtriangledown u)^2 \geq \mu q (1-u)^{-(q+1)} J.$$

Then, by the maximum principle (cf. Protter and Weinberger [6, p. 190]), J and H are nonnegative on $\bar{\Omega}'$. Therefore, when $v \to \infty$, $u_t \to \infty$. Similarly, $v_t \to \infty$ when $u \to 1^-$. This completes the proof. □

Let us denote $\phi(x) > 0$ in D and $\lambda_1 > 0$ respectively, the first eigenfunction and the corresponding eigenvalue of the following Sturm-Liouville problem,

$$\Delta z + \lambda z = 0 \text{ in } D,\ z = 0 \text{ on } \partial D.$$

We choose the function $\phi(x)$ such that $||\phi||_{L^1(D)} = 1$.

Lemma 2.3 *The solution u of the problem (1.1)-(1.2) reaches 1 somewhere in D in a finite time T if and only if the solution v tends to infinity somewhere in D in the finite time T.*

Proof. Suppose that u reaches 1 somewhere in D in a finite time T, and that v is bounded by a positive constant M on $\bar{\Omega}$. By Equation (3.2.21) of Friedman [2, p. 65], the Hölder norm of u satisfies

$$|u|_{C^{2+\alpha,1+\alpha/2}(\Omega)} \leq k_0 \left(|u_0|_{C^{2+\alpha}(\Omega)} + \gamma M^P \right),$$

where k_0 is a positive constant. Therefore, u_t is bounded, say by k_1, in Ω. It follows from the estimation in the proof of Lemma 2.2 that $k_1 v_t \geq u_t v_t \geq \kappa\mu / (1-u)^q u_t$. Upon integration from 0 to t, we obtain

$$k_1(M - v_0) \geq k_1(v(x,t) - v_0) \geq \kappa\mu \left((1-u(x,t))^{-q+1} - (1-u_0(x))^{-q+1} \right) / (q-1).$$

Since u reaches 1 somewhere in D as $t \to T$, and $q > 1$, we have a contradiction. This shows that v is unbounded in Ω.

Suppose that v tends to infinity somewhere in D as $t \to T$, and assume that $u \leq \delta < 1$ on $\bar{\Omega}$. Then, $v_t - \Delta v \leq \mu / (1-\delta)^q = M_2$ for some positive constant M_2, and this gives $v(x,t) \leq M_2 e^{\lambda_1 t}$ for all t. This contradicts with v is unbounded in the finite time T. □

Let $R(t) = \int_D u\phi dx$, $S(t) = \int_D v\phi dx$, and $E(t) = R(t) + S(t)$. We modify the method of Samarkii, Galaktionov, Kurdymov, and Mikhailov [7, pp. 447-453] to show that v blows up and u quenches simultaneously in a finite time.

Theorem 2.4 *(a) If $p = 1$ and $\eta > \lambda_1$ in which $\eta = \min\{\gamma, \mu q\}$, then v blows up and u quenches simultaneously in a finite time.*

(b) If $2 > p > 1$ and $E(0) \geq E^*$ where E^* is the first positive root of the equation

$$\gamma 2^{-p} E^p - \lambda_1 E - \frac{(2-p)\,\mu q\,(q+1)}{4} \left(\frac{p}{2}\right)^{p/(2-p)} = 0,$$

then v blows up and u quenches simultaneously in a finite time.

(c) If $p = 2$ and $\gamma E(0)/2 > \lambda_1$, then v blows up and u quenches simultaneously in a finite time.

(d) If $p > 2$ and $E(0) \geq \left(\lambda_1 + \sqrt{\lambda_1^2 + \gamma\beta}\right) 2/\gamma$ where

$$\beta = \gamma\,(2/p)^{2/(p-2)}\,(p-2)/p,$$

then v blows up and u quenches simultaneously in a finite time.

Proof. Multiply $\phi(x)$ on the both sides of the equations (1.1) and integrate the expressions over the domain D, we obtain

$$\int_D u_t \phi dx - \int_D \Delta u \phi dx = \gamma \int_D v^p \phi dx, \tag{2.2}$$

$$\int_D v_t \phi dx - \int_D \Delta v \phi dx = \mu \int_D \frac{1}{(1-u)^q} \phi dx. \tag{2.3}$$

Then, by using the Green's second identity and the boundary condition, the equations (2.2) and (2.3) become

$$\left(\int_D u \phi dx \right)_t = -\lambda_1 \int_D u \phi dx + \gamma \int_D v^p \phi dx, \tag{2.4}$$

$$\left(\int_D v \phi dx \right)_t = -\lambda_1 \int_D v \phi dx + \mu \int_D \frac{1}{(1-u)^q} \phi dx. \tag{2.5}$$

According to the McClaurin series, for $0 \leq u < 1$, we have

$$\frac{1}{(1-u)^q} = 1 + qu + \frac{q(q+1)}{2} u^2 + \cdots \geq qu, \tag{2.6}$$

or

$$\frac{1}{(1-u)^q} \geq \frac{q(q+1)}{2} u^2. \tag{2.7}$$

For $p \geq 1$, by the Jensen's inequality, the equation (2.4) becomes

$$\left(\int_D u \phi dx \right)_t \geq -\lambda_1 \int_D u \phi dx + \gamma \left(\int_D v \phi dx \right)^p. \tag{2.8}$$

It follows from (2.6) and (2.5)

$$\left(\int_D v \phi dx \right)_t \geq -\lambda_1 \int_D v \phi dx + \mu q \int_D u \phi dx dt. \tag{2.9}$$

By (2.7), (2.5), and the Jensen's inequality, we obtain

$$\left(\int_D v \phi dx \right)_t \geq -\lambda_1 \int_D v \phi dx + \frac{\mu q (q+1)}{2} \left(\int_D u \phi dx \right)^2. \tag{2.10}$$

(a) When $p = 1$, we add the inequalities (2.8) and (2.9), and obtain

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \gamma S + \mu q R.$$

Let $\eta = \min\{\gamma,\ \mu q\}$. The above expression is equivalent to

$$\frac{dE}{dt} \geq (\eta - \lambda_1) E.$$

Assume that $\eta > \lambda_1$. Then, integrate the above inequality from 0 to t, we have

$$E(t) \geq E(0) e^{(\eta - \lambda_1)t}.$$

On the right hand side of this inequality, t indicates existence time of $E(t)$.

Suppose that $E(t)$ exists for all $t > 0$, then $E(t) \to \infty$ as $t \to \infty$. Since $R(t) \leq 1$, there exists $t_0 > 0$ such that $S(t) > 1$ for $t \geq t_0$. It follows from the inequality (2.4) that $\frac{dR}{dt} \geq -\lambda_1 R + \gamma S > -\lambda_1 S + \eta S = (\eta - \lambda_1) S$ for $t \geq t_0$. Integrate this inequality from t_0 to t, we have $R(t) \geq R(t_0) + (\eta - \lambda_1) S(t_0)(t - t_0)$. This implies that $R(t)$ reaches 1 in a finite time T, and this contradicts with the global existence of $E(t)$.

This gives $v \to \infty$ or $u \to 1^-$ in a finite time T. It follows from the Lemma 2.3 that v blows up and u quenches simultaneously in a finite time.

(b) When $2 > p > 1$, by the inequalities (2.8) and (2.10), it yields

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \gamma S^p + \frac{\mu q (q+1)}{2} R^2. \tag{2.11}$$

For any $\rho > \nu$, by the Young's inequality, we have

$$\frac{1}{\frac{1+\rho}{1+\nu}} \left(R^{1+\nu}\right)^{(1+\rho)/(1+\nu)} + \frac{1}{\frac{1+\rho}{\rho-\nu}} \left(\frac{1+\nu}{1+\rho}\right)^{(1+\rho)/(\rho-\nu)} \geq \frac{1+\nu}{1+\rho} R^{1+\nu}.$$

In particular, take $\nu = p-1$ and $\rho = 1$, we obtain

$$\frac{p}{2} R^2 + \frac{2-p}{2} \left(\frac{p}{2}\right)^{2/(2-p)} \geq \frac{p}{2} R^p,$$

it gives

$$R^2 \geq R^p - \frac{(2-p)}{2} \left(\frac{p}{2}\right)^{p/(2-p)}.$$

Consequently, the inequality (2.11) becomes

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \gamma S^p + \frac{\mu q (q+1)}{2} \left[R^p - \frac{(2-p)}{2} \left(\frac{p}{2}\right)^{p/(2-p)} \right].$$

Assume that $\gamma \leq \mu q (q+1)/2$, we get

$$\begin{aligned} &\frac{d}{dt}(R+S) \\ \geq\ & -\lambda_1 (R+S) + \gamma S^p \left[1 + \frac{\frac{\mu q (q+1)}{2} R^p}{\gamma S^p}\right] - \frac{(2-p)\, \mu q (q+1)}{4} \left(\frac{p}{2}\right)^{p/(2-p)} \\ \geq\ & -\lambda_1 (R+S) + \gamma S^p \left(1 + \frac{R^p}{S^p}\right) - \frac{(2-p)\, \mu q (q+1)}{4} \left(\frac{p}{2}\right)^{p/(2-p)}. \end{aligned}$$

By using the property of convex functions, we have

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \gamma S^p 2^{-(p-1)} \left(1 + \frac{R}{S}\right)^p - \frac{(2-p)\, \mu q (q+1)}{4} \left(\frac{p}{2}\right)^{p/(2-p)}.$$

Equivalently,

$$\frac{d}{dt}E \geq -\lambda_1 E + \gamma 2^{-(p-1)} E^p - \frac{(2-p)\,\mu q\,(q+1)}{4}\left(\frac{p}{2}\right)^{p/(2-p)}.$$

By Lemma 2.1, u_t and v_t are positive in Ω. Thus, E is an increasing function in t. Let E^* be the first positive root of the equation

$$\gamma 2^{-p} E^p - \lambda_1 E - \frac{(2-p)\,\mu q\,(q+1)}{4}\left(\frac{p}{2}\right)^{p/(2-p)} = 0.$$

Assume that $E\,(0) \geq E^*$, we have

$$\frac{d}{dt}E \geq \gamma 2^{-p} E^p.$$

This shows that $E\,(t)$ exists in a finite time only. Use the similar argument as in the proof of (a), we conclude that v blows up and u quenches simultaneously in a finite time.

A similar result can be obtained for $\gamma > \mu q\,(q+1)\,/2$.

(c) When $p = 2$, by the inequality (2.11) we get

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \gamma S^2 + \frac{\mu q\,(q+1)}{2} R^2.$$

Assume that $\gamma \leq \mu q\,(q+1)\,/2$, the reverse case can be proved similarly, then,

$$\begin{aligned}
\frac{d}{dt}(R+S) &\geq -\lambda_1 (R+S) + \gamma S^2 + \frac{\mu q\,(q+1)}{2} R^2 \\
&\geq -\lambda_1 (R+S) + \gamma S^2 \left(1 + \frac{R^2}{S^2}\right) \\
&= -\lambda_1 (R+S) + \frac{\gamma}{2}(R+S)^2.
\end{aligned}$$

This is equivalent to

$$\frac{d}{dt}E \geq -\lambda_1 E + \frac{\gamma}{2}E^2.$$

Assume that $\gamma E\,(0)\,/2 > \lambda_1$, we have

$$\int_{E(0)}^{E(t)} \frac{dE}{E\left(-\lambda_1 + \frac{\gamma}{2}E\right)} \geq \int_0^t dt.$$

After integration, it gives

$$\frac{1}{\lambda_1}\ln\left[\frac{E\,(0)\left(\frac{\gamma}{2}E\,(t) - \lambda_1\right)}{E\,(t)\left(\frac{\gamma}{2}E\,(0) - \lambda_1\right)}\right] \geq t.$$

Therefore, $E(t)$ exists only in a finite time. Hence, use the similar argument as in the proof of (a), we conclude that v blows up and u quenches simultaneously in a finite time.

(d) When $p > 2$, from the inequality (2.11), we get

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \gamma S^p + \frac{\mu q(q+1)}{2} R^2.$$

Using the Young's inequality, it gives

$$S^p \geq S^2 - \frac{(p-2)}{p}\left(\frac{2}{p}\right)^{2/(p-2)}.$$

Consequently,

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \gamma \left[S^2 - \frac{(p-2)}{p}\left(\frac{2}{p}\right)^{2/(p-2)}\right] + \frac{\mu q(q+1)}{2} R^2.$$

Assume that $\gamma \leq \mu q(q+1)/2$, the calculation is similar for the reverse case, then

$$\begin{aligned}\frac{d}{dt}(R+S) &\geq -\lambda_1 (R+S) + \gamma S^2 \left[1 + \frac{\frac{\mu q(q+1)}{2} R^2}{\gamma S^2}\right] - \frac{\gamma(p-2)}{p}\left(\frac{2}{p}\right)^{2/(p-2)} \\ &\geq -\lambda_1 (R+S) + \gamma S^2 \left(1 + \frac{R^2}{S^2}\right) - \frac{\gamma(p-2)}{p}\left(\frac{2}{p}\right)^{2/(p-2)}.\end{aligned}$$

By using the property of convex functions, we get

$$\frac{d}{dt}(R+S) \geq -\lambda_1 (R+S) + \frac{\gamma}{2}(S+R)^2 - \frac{\gamma(p-2)}{p}\left(\frac{2}{p}\right)^{2/(p-2)}.$$

Let $\beta = \gamma(2/p)^{2/(p-2)}(p-2)/p$. The above inequality is equivalent to

$$\frac{d}{dt}E \geq -\lambda_1 E + \frac{\gamma E^2}{2} - \beta.$$

The positive root of the quadratic equation

$$\frac{\gamma}{4}E^2 - \lambda_1 E - \beta = 0,$$

is given by

$$E = \frac{\lambda_1 + \sqrt{\lambda_1^2 - 4\left(\frac{\gamma}{4}\right)(-\beta)}}{2\left(\frac{\gamma}{4}\right)} = \frac{\lambda_1 + \sqrt{\lambda_1^2 + \gamma\beta}}{\frac{\gamma}{2}}.$$

Now, by assumption $E(0) \geq \left(\lambda_1 + \sqrt{\lambda_1^2 + \gamma\beta}\right) 2/\gamma$, we obtain $\frac{d}{dt}E \geq \frac{\gamma E^2}{4}$. By the similar argument as in the proof of (a), v blows up and u quenches simultaneously in a finite time. □

3 Quenching and Blow-up Rate

In this section, we are going to give the quenching and blow-up rate of the problem (1.1)-(1.2). We need the following lemmas regarding the system of differential inequalities.

Lemma 3.1 *Let $p \geq 1$, $q > 1$, ϵ_1 and ϵ_2 be positive real numbers, and $y(t)$ and $z(t)$ be positive functions satisfying the inequalities*

$$\left.\begin{array}{ll} y'(t) & \geq \epsilon_1 z^p, \\ z'(t) & \geq \epsilon_2 (1-y)^{-q}, \end{array}\right\} \tag{3.1}$$

with $0 < y(0) < 1$ and $z(0) > 0$ for $0 < t < T$. Assume that $y(t) \to 1$ and $z(t) \to \infty$ as $t \to T$, then there exist positive constants K_1 and K_2 such that $y(t) \leq 1 - K_1(T-t)^{(p+1)/(pq+1)}$ and $z(t) \leq K_2(T-t)^{-(q+1+2pq)/(pq+1)}$.

Proof. Firstly, let us look for an upper bounded for $y(t)$. Suppose that $y(t) \leq 1 - K_1(T-t)^{(p+1)/(pq+1)}$ does not hold for any positive constant K_1, then $\underline{\lim}_{t\to T}(1-y(t))(T-t)^{-(p+1)/(pq+1)} = 0$. This implies that there exist an increasing sequence $\{t_i\}_{i=1}^{\infty}$ with $t_i \to T$ and a decreasing sequence $\{\eta_i\}_{i=1}^{\infty}$ with $\eta_i \to 0$ such that $1 - y(t_i) < \eta_i (T-t_i)^{(p+1)/(pq+1)}$.

For fixed i, when $t \in (0, T-t_i)$, by integrating the second inequality of (3.1), we get

$$\begin{aligned} z(t_i+t) - z(t_i) &\geq \epsilon_2 \int_{t_i}^{t_i+t} (1-y(s))^{-q} ds \\ &\geq \epsilon_2 \eta_i^{-q} (T-t_i)^{-q(p+1)/(pq+1)} t. \end{aligned}$$

This gives

$$\begin{aligned} y'(t_i+t) &\geq \epsilon_1 \left[z(t_i) + \epsilon_2 \eta_i^{-q} (T-t_i)^{-q(p+1)/(pq+1)} t \right]^p \\ &\geq \epsilon_1 \left[\epsilon_2 \eta_i^{-q} (T-t_i)^{-q(p+1)/(pq+1)} t \right]^p. \end{aligned}$$

Upon integration for t from 0 to $T - t_i$, we have

$$y(t_i + (T-t_i)) - y(t_i) \geq \epsilon_1 \left[\epsilon_2 \eta_i^{-q} (T-t_i)^{-q(p+1)/(pq+1)} \right]^p \frac{1}{p+1} (T-t_i)^{p+1}.$$

Hence, $1 - y(t_i) \geq \frac{1}{p+1} \epsilon_1 \epsilon_2^p \eta_i^{-pq} (T-t_i)^{(p+1)/(pq+1)}$. By using $1 - y(t_i) < \eta_i (T - t_i)^{(p+1)/(pq+1)}$, we have $\eta_i^{1+pq} > \frac{1}{p+1} \epsilon_1 \epsilon_2^p$ for any i. Since $\eta_i \to 0$, we have a contradiction. This shows that $y(t) \leq 1 - K_1(T-t)^{(p+1)/(pq+1)}$ for some positive constant K_1.

To find an upper bound of $z(t)$, we integrate the second inequality of (3.1) and get $z(t) \geq z(0) + \epsilon_2 \int_0^t (1-y(s))^{-q} ds \geq \epsilon_2 \int_0^t (1-y(s))^{-q} ds$. From $y(t) \geq y(0) + \epsilon_1 \int_0^t z^p(\sigma) d\sigma$, we have $z(t) \geq \epsilon_2 \int_0^t (1 - \epsilon_1 \int_0^s z^p(\sigma) d\sigma)^{-q} ds$.

Let $k(t)=\epsilon_2\int_0^t(1-\epsilon_1\int_0^s z^p(\sigma)d\sigma)^{-q}ds$. Then $k(t)\le z(t)$, $k(0)=0$, and $k'(t)\ge\epsilon_2>0$ for $t\ge0$. By a direct computation, we get

$$\begin{aligned}
& [(k'(t))^{(q-1)/q}]' \\
= \; & \frac{q-1}{q}(k'(t))^{-1/q}k''(t) \\
= \; & \frac{q-1}{q}\epsilon_2^{-1/q}\left(1-\epsilon_1\int_0^t z^p(\sigma)d\sigma\right)\epsilon_2(-q)\left(1-\epsilon_1\int_0^t z^p(\sigma)d\sigma\right)^{-q-1}(-\epsilon_1 z^p(t)) \\
= \; & (q-1)\epsilon_2^{-1/q}\epsilon_1\epsilon_2\left(1-\epsilon_1\int_0^t z^p(\sigma)d\sigma\right)^{-q}z^p(t) \\
\ge \; & (q-1)\epsilon_2^{-1/q}\epsilon_1 k'(t)k^p(t)=\frac{q-1}{p+1}\epsilon_2^{-1/q}\epsilon_1\left(k(t)^{p+1}\right)'.
\end{aligned}$$

Upon integration, we have $(k'(t))^{(q-1)/q}-(k'(0))^{(q-1)/q}\ge\frac{q-1}{p+1}\epsilon_2^{-1/q}\epsilon_1 k^{p+1}(t)$. Since $k'(0)>0$, we have $(k'(t))^{(q-1)/q}\ge\frac{q-1}{p+1}\epsilon_2^{-1/q}\epsilon_1 k^{p+1}(t)$. This implies $k(t)^{-\frac{q(p+1)}{q-1}}k'(t)\ge\left(\frac{q-1}{p+1}\epsilon_2^{-1/q}\epsilon_1\right)^{q/(q-1)}$. Integrate from t to T, we get

$$\frac{q-1}{pq+1}\left[k(t)^{-\frac{pq+1}{q-1}}-k(T)^{-\frac{pq+1}{q-1}}\right]\ge\left(\frac{q-1}{p+1}\epsilon_2^{-1/q}\epsilon_1\right)^{q/(q-1)}(T-t).$$

Since $k(T)>0$, we obtain $k^{-\frac{pq+1}{q-1}}(t)\ge\frac{pq+1}{q-1}\left(\frac{q-1}{p+1}\epsilon_2^{-1/q}\epsilon_1\right)^{q/(q-1)}(T-t)$. This implies $k(t)\le K(T-t)^{-\frac{q-1}{pq+1}}$ for some positive constant K.

It follows from $z(t)\to\infty$ as $t\to T$, for some fixed $t<T$, we have $z(t)>1$. Consider $\tau=(T-t)/4$, we have

$$\begin{aligned}
k(t+2\tau) &= \epsilon_2\int_0^{t+2\tau}\left(1-\epsilon_1\int_0^s z^p(\sigma)d\sigma\right)^{-q}ds \\
&\ge \epsilon_2\int_{t+\tau}^{t+2\tau}\left(1-\epsilon_1\int_0^s z^p(\sigma)d\sigma\right)^{-q}ds \\
&\ge \epsilon_2\left(1-\epsilon_1\int_0^{t+\tau}z^p(\sigma)d\sigma\right)^{-q}\tau \\
&\ge \epsilon_2\left(1+\epsilon_1 q\int_0^{t+\tau}z^p(\sigma)d\sigma\right)\tau \\
&\ge \epsilon_1\epsilon_2 q\tau\int_t^{t+\tau}z^p(\sigma)d\sigma \\
&\ge \epsilon_1\epsilon_2 q\tau^2 z^p(t)\ge\epsilon_1\epsilon_2 q\tau^2 z(t).
\end{aligned}$$

Hence, $K(T-t-2\tau)^{-\frac{q-1}{pq+1}}\ge k(t+2\tau)\ge\epsilon_1\epsilon_2 q\tau^2 z(t)$. By using $T-t-2\tau=(T-t)/2$, we have $2^{\frac{q-1}{pq+1}}K(T-t)^{-\frac{q-1}{pq+1}}\ge\epsilon_1\epsilon_2 q\,(T-t)^2 z(t)/16$, that is $z(t)\le K_2(T-t)^{-\frac{(q+1+2pq)}{pq+1}}$, for some constant $K_2>0$. □

On the other hand, we have the following estimation.

Lemma 3.2 *Let $p \geq 1$, $q > 1$, ν_1 and ν_2 be positive real numbers, and $y(t)$ and $z(t)$ be positive functions satisfying the inequalities*

$$\left.\begin{array}{ll} y'(t) & \leq \nu_1 z^p, \\ z'(t) & \leq \nu_2 (1-y)^{-q}, \end{array}\right\} \tag{3.2}$$

with $0 < y(0) < 1$ and $z(0) > 0$ for $0 < t < T$. Assume that $y(t) \to 1$ and $z(t) \to \infty$ as $t \to T$, then there exist positive constants ς and β such that the functions $\bar{y}(t) = 1 - \varsigma(T-t)^{\frac{p+1}{pq+1}}$ and $\bar{z}(t) = \beta(T-t)^{-\frac{q-1}{pq+1}}$ satisfy $y(t) \geq \bar{y}(t)$, and $z(t) \geq \bar{z}(t)$ for $T > t \geq t_0 > 0$ for some fixed $t_0 < T$.

Proof. Let $\bar{y}(t) = 1 - \varsigma(T-t)^{\frac{p+1}{pq+1}}$ and $\bar{z}(t) = \beta(T-t)^{-\frac{q-1}{pq+1}}$ where ς and β are positive constants. Then $\bar{y}'(t) = \varsigma\left(\frac{p+1}{pq+1}\right)(T-t)^{-\frac{p(q-1)}{pq+1}}$ and $\bar{z}'(t) = \beta\left(\frac{q-1}{pq+1}\right)(T-t)^{-\frac{q(p+1)}{pq+1}}$. Let ν_1 and ν_2 be positive constants such that $\nu_1 = \varsigma\left(\frac{p+1}{pq+1}\right)\frac{1}{\beta^p}$, and $\nu_2 = \beta\left(\frac{q-1}{pq+1}\right)\varsigma^q$, then $\bar{y}'(t) = \nu_1 \bar{z}^p(t)$, and $\bar{z}'(t) = \nu_2(1-\bar{y})^{-q}$. Let $t_0 \in (0,T)$ such that $0 < \bar{y}(t_0)$.

If there exists some $t_1 \in [t_0, T)$ such that $\bar{y}(t_1) > y(t_1)$ and $\bar{z}(t_1) > z(t_1)$, by $\bar{y}'(t) > 0$, and $\bar{z}'(t) > 0$, there is an $\eta > 0$ such that $\bar{y}(t_1 - \eta) > y(t_1)$, and $\bar{z}(t_1 - \eta) > z(t_1)$. It follows from (3.2) and the comparison result that $\bar{y}(t-\eta) > y(t)$ and $\bar{z}(t-\eta) > z(t)$ for $t \in (t_1, T)$. Hence, $1 > \bar{y}(T-\eta) > y(T) = 1$. It leads to a contradiction. This shows that either $\bar{y}(t) \leq y(t)$ or $\bar{z}(t) \leq z(t)$ in $[t_0, T)$.

Assume that $\bar{y}(t) \leq y(t)$ in $[t_0, T)$, and $z(t)(T-t)^{\frac{q-1}{pq+1}}$ is not bounded away from zero in $[t_0, T)$. Then there exist sequences $\{\eta_i\}$ and $\{t_i\}$ such that $\eta_i \to 0$, $t_i \to T$ and $z(t_i) < \eta_i (T-t_i)^{-\frac{q-1}{pq+1}}$. It follows from the Lemma 3.1 that there is a positive constant C such that $y(t) \leq 1 - C(T-t)^{\frac{p+1}{pq+1}}$. Take $\kappa > 0$ such that $C(\kappa+1)^{\frac{p+1}{pq+1}} > \varsigma$, let $t_i' = t_i - \kappa(T-t_i)$. Then $T - t_i' = (1+\kappa)(T-t_i)$, $t_i' < t_i$, and $t_i' \to T$ as $i \to \infty$.

By using $1 - \varsigma(T-t_i)^{\frac{p+1}{pq+1}} = \bar{y}(t_i) \leq y(t_i)$ for $t_i \in [t_0, T)$, $y(t) \leq 1 - C(T-t)^{\frac{p+1}{pq+1}}$, and $\int_{t_i'}^{t_i} \nu_1 z^p(t)dt \geq y(t_i) - y(t_i')$, we obtain

$$\begin{aligned} 1 - \varsigma(T-t_i)^{\frac{p+1}{pq+1}} &\leq y(t_i') + \nu_1 z^p(t_i)(t_i - t_i') \\ &\leq 1 - C(T-t_i')^{\frac{p+1}{pq+1}} + \nu_1 \eta_i^p (T-t_i)^{-\frac{(q-1)p}{pq+1}} \kappa(T-t_i). \end{aligned}$$

This gives $C(\kappa+1)^{\frac{p+1}{pq+1}} \leq (\varsigma + \nu_1 \eta_i^p)$. Hence $C(\kappa+1)^{\frac{p+1}{pq+1}} \leq \varsigma$ when $i \to \infty$. It contradicts to the choice of κ. This implies that $\bar{z}(t) \leq z(t)$ if $\bar{y}(t) \leq y(t)$ for $t \in [t_0, T)$. Similarly, we have $\bar{y}(t) \leq y(t)$ if $\bar{z}(t) \leq z(t)$ for $t \in [t_0, T)$. □

Theorem 3.3 *Let $p \geq 1$, $q > 1$, and (u,v) be the solution of the problem (1.1)-(1.2) with (u_0, v_0) satisfying the inequalities (2.1). Assume that there* *exists $T < \infty$ such that u quenches and v blows up as $t \to T^-$. Then there exist positive constants C_1, C_2, C_3, and C_4 such that*

$$\left.\begin{array}{rcl} C_1(T-t)^{(p+1)/(pq+1)} \leq & 1-\|u(t)\|_\infty & \leq C_3(T-t)^{(p+1)/(pq+1)}, \\ C_2(T-t)^{-(q-1)/(pq+1)} \leq & \|v(t)\|_\infty & \leq C_4(T-t)^{-(q+1+2pq)/(pq+1)}. \end{array}\right\}$$

Proof. It follows a similar argument as in the proof of the Lemma 2.2 that there exists $\kappa > 0$ such that $u_t \geq \kappa\gamma v^p$, and $v_t \geq \kappa\mu(1-u)^{-q}$ on $\bar{D}' \times [0,T)$ where $D' \subset D$. Hence, by the Lemma 3.1, there exist positive constants C_1 and C_2 such that $u(x,t) \leq 1 - C_1(T-t)^{(p+1)/(pq+1)}$ and $v(x,t) \leq C_2(T-t)^{-(q-1)/(pq+1)}$ on $\bar{D}'$. Since u and v do not quench or blow up in $D\backslash D'$, we have $\|u(x,t)\|_\infty \leq 1 - C_1(T-t)^{(p+1)/(pq+1)}$ and $\|v(x,t)\|_\infty \leq C_4(T-t)^{-(q+1+2pq)/(pq+1)}$.

Furthermore, let $U(t) = \max_{x\in\bar{D}} u(x,t)$, and $V(t) = \max_{x\in\bar{D}} v(x,t)$. Then we obtain $U'(t) \leq \gamma V^p(t)$ and $V'(t) \leq \mu(1-U(t))^{-q}$. It follows from the Lemma 3.2 that $U(t) \geq 1 - C_3(T-t)^{(p+1)/(pq+1)}$ and $V(t) \geq C_2(T-t)^{-(q-1)/(pq+1)}$. This gives $1 - C_3(T-t)^{(p+1)/(pq+1)} \geq \|u(x,t)\|_\infty$ and $\|v(x,t)\|_\infty \geq C_2(T-t)^{-(q-1)/(pq+1)}$. This completes the proof. □

References

[1] K. Deng and H. A. Levine, On the blowup of u_t at quenching, Proc. Amer. Math. Soc., 106 (1989), 1049-1056.

[2] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, NJ, 1964, p. 65.

[3] A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J., 34 (1985), 425-447.

[4] J-S Guo, S. Sasayama, and C-J Wang, Blow-up rate estimate for a system of semilinear parabolic equations, Comm. Pure Appl. Anal., 8 (2009), 711-718.

[5] C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, NY, 1992, pp. 383-384, 393, 436, and 687.

[6] M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, NJ, 1967, p. 190.

[7] A. A. Samarkii, V. A. Galaktionov, S. P. Kurdymov, and A. P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin, 1995, pp. 447-453.

[8] L. Wang, The blow-up for weakly coupled reaction-diffusion systems, Proc. Amer. Math. Soc., 129 (2001), 89-95.

Received: October, 2010