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A Comparison between Solving Two Dimensional Integral Equations by the Traditional Collocation Method and Radial Basis Functions

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Abstract

In this paper, we discuss about the methods which have spectral accuracy for solving two dimensional Fredholm integral equations. At first, we introduce the traditional spectral methods (collocation , Tau and Galerkin method), and then we discussed about radial basis functions (RBF) method. Since, radial basis functions produce exponential convergence (spectral convergence) [15], this method is compared with the traditional collocation method by the orthogonal polynomials. Numerical results show that RBF method is faster than the conventional spectral methods. It is conform with our expectation, because radial basis functions are extraordinary powerful in higher dimensions.

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1. Introduction

Two dimensional integral equations provide an important tool for modeling a numerous problems in engineering and mechanics [2, 11]. There are many different numerical methods for solving integral equations. Some of them can be used for solving double integral equations. Computational complexity of mathematical operations is the most important obstacle for solving integral equations in higher dimensions.

The Nystrom method [10] and collocation method [7, 9, 18] are the most important approaches of the numerical solution of these integral equations. Recently, differential transform, radial basis functions and new other methods are applied for solving two dimensional linear and nonlinear integral equations [1, 13, 17].

In this study, we investigate the spectral methods and RBF method for solving double integral equations. hence, in section 2, we explain spectral methods generally. In section 3, radial basis functions are illustrated. In section 4, we explain the mentioned methods for solving two dimensional integral equation summarily. Also, some examples in section 5 show good results for all of the mentioned methods. In section 6, We recommend you the use of RBF method in higher dimensions.

2. Spectral method

The most common methods are based on approximation by orthogonal basis. In fact, every continuous function can be approximated by orthogonal polynomials [4, 12]. For example $f(x)$ can be rewritten in the following form

$$f(x) = \sum_{i=0}^{n} a_i \phi_i(x), \tag{1}$$

where n is natural number and $\phi_i(x), i = 0, 1, ..., n$ are orthogonal basis. There are different family of orthogonal functions such as Chebyshev polynomials, Legendre polynomials, etc. These approximations are valid in special intervals. In this work, we apply Chebyshev polynomials.

When $f(x)$ is unknown in a equation, we substitute (1) instead of it. Now we must determine $a_i, i = 1, 2, ..., n$. If we use inner product property, It lead us to Tau method. Also, if we replacing some suitable point in (1), It lead us collocation method. Galerkin method is similar Tau method except a few differences in choosing of basis such that satisfy boundary conditions.

These methods provide exponential convergence to exact solution [6], hence we call them spectral methods. These methods are powerful tools for solving of different kinds of differential and integral equations. Also, these methods are developed for higher dimensions as partial differential equation (PDE) and double integral equations. In this study, we just discuss about two dimensional integral equations.

3. Radial basis functions

Definition(Basic RBF Method): Given a set of n distinct data points $\{p_j\}_{j=0}^{n}$ and corresponding data value $\{f_j\}_{j=0}^{n}$, $f(x)$ can be written the following form

$$\Phi(p) = \sum_{j=0}^{n} \lambda_j \phi(\|p - p_j\|), \tag{2}$$

where $\|.\|$ is Euclidean norm, $p, p_j \in \mathbb{R}^d$(d is positive finite integer) and f_j is scalar. Also $\phi(r), r >= 0$, is some radial basis functions. The coefficient λ_j must be determined as unknown. Collocating of $\{x_j\}_{j=0}^{n}$ in (2) leads us to following symmetric linear system

$$\begin{bmatrix} \\ \quad A \quad \\ \\ \end{bmatrix} \begin{bmatrix} \\ \lambda \\ \\ \end{bmatrix} = \begin{bmatrix} \\ f \\ \\ \end{bmatrix}, \tag{3}$$

where the entries of A are given by

$$a_{jk} = \phi(\|p_j - p_k\|). \tag{4}$$

Micchelli [16] gave sufficient conditions for $\phi(r)$ in (3) to guarantee that A matrix in (4) is unconditionally nonsingular, and thus that the basic RBF method is uniquely solvable. Some common infinitely smooth example of the $\phi(r)$ that lead to a uniquely solvable method are the following forms

Linear	r
Gaussian(GA)	$e^{-(\epsilon r)^2}$
Multi Quadric(MQ)	$(1 + (\epsilon r)^2)^{\frac{\alpha}{2}} \quad (\alpha \neq 0, \alpha \neq 2\mathbb{N})$
Inverse Multi Quadric(IMQ)	$(1 + (\epsilon r)^2)^{-\frac{1}{2}}$
Inverse Quadric(IQ)	$(1 + (\epsilon r)^2)^{-1}$.

Parameter ϵ is a free parameter for controlling the shape of functions. Also, there are some piecewise smooth RBF such as r^3(Cubic) and $r^2 \log r$(Thin plate spline).

It has been discussed about sufficient conditions for $\phi(r)$ to guarantee nonsingularity of the A matrix [5, 8]. These conditions show that a larger class of functions could be considered. The most important privilege of this method is exponentially convergence, however discussion about convergence is more complicated for these class of functions.

4. Solution of linear two dimensional integral equation

Consider the two dimensional linear Fredholm integral equation as follows

$$u(x,t) - \int_{-1}^{1}\int_{-1}^{1} k(x,t,y,z)u(y,z)dydz = f(x,t), \qquad (x,t) \in [-1,1] \times [-1,1], \tag{5}$$

where $k(x,t,y,z)$ and $f(x,t)$ are continuous functions on $[-1,1]^4$ and $[-1,1]^2$ respectively. For the case which integration domain is $[a,b]\times[c,d]$, we can use suitable change of variable to obtain this intervals.

4.1. Solving by Chebyshev polynomials

First we solve two dimensional integral equation by Chebyshev polynomials [4, 14]. Function $u(x,t)$ defined over $[-1,1]\times[-1,1]$ may be represented by Chebyshev series as

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}T_i(x)T_j(t), \qquad (x,t)\in[-1,1]\times[-1,1]. \tag{6}$$

If the infinite series in (6) is truncated, then it can be written as

$$u(x,t)\approx u_N(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}T_i(x)T_j(t), \tag{7}$$

where N is any natural number. Also, we let

$$R_N(x,t)=u_N(x,t)-\int_{-1}^{1}\int_{-1}^{1}k(x,t,y,z)u_N(y,z)dydz-f(x,t), \tag{8}$$

The method of collocation solves the (5) using the approximation (7) through the equations

$$R_N(x_r,t_s)=u_N(x_r,t_s)-\int_{-1}^{1}\int_{-1}^{1}k(x_r,t_s,y,z)u_N(y,z)dydz-f(x_r,t_s)=0, \tag{9}$$

for collocation points

$$\begin{cases} x_r=\cos(\frac{r\pi}{N}), & r=0,1,...,N,\\ t_s=\cos(\frac{s\pi}{N}), & s=0,1,...,N. \end{cases} \tag{10}$$

The Tau method solves (5) using property of inner product that leads the following equations

$$<R_N(x,t),T_r(x)T_s(t)>=0, \qquad r,s=0,1,...,N. \tag{11}$$

This method can be provided by Legendre polynomials similarly.

4.1. Solving by radial basis functions

Consider (5) again or furthurmore, we can consider the more general form as follow

$$u(x,y)-\int_c^d\int_a^b G(x,y,s,t,u(s,t))dsdt=f(x,y),\qquad (x,y)\in[a,b]\times[c,d],\quad(12)$$

where $G(x,y,s,t,u(s,t))$ and $f(x,y)$ are given analytic functions when $a\leq x,y,s,t\leq b$.

Recently, these equations solved by RBF [1]. According to (2), the function $u(x,y)$ may be represented by approximate series as

$$u(p)\backsimeq\sum_{\gamma=0}^{n}c_\gamma\phi(\|p-p_\gamma\|)=C^T\Psi(p),\qquad(13)$$

where n is any natural number, p and p_γ are points of $\mathbb{R}^2$ as (x,y) and (x_i,y_j) respectively. By previous section, it is clear that $\{p_\gamma\}_{\gamma=0}^{n}$ can be given arbitrary as centers. However, the selection process of the center points effects on accuracy, sometimes uniform points or random points are preferred.

With replacing (13) in (12) we have

$$C^T\Psi(x,y)-\int_c^d\int_a^b G(x,y,s,t,C^T\Psi(s,t))dsdt=f(x,y),\qquad (x,y)\in[a,b]\times[c,d],\qquad(14)$$

In the above equation, only c_γ $(\gamma=0,1,...,n)$ are unknowns and it is the interesting technical advantage in using of RBFs. It means the process of solving is no more complicated in spite of increasing the dimension of problem.

Now we substitute the given collocation points in the above equation. The collocation points can be the same center points or any other arbitrary points.

$$C^T\Psi(x_i,y_j)-\int_c^d\int_a^b G(x_i,y_j,s,t,C^T\Psi(s,t))dsdt=f(x_i,y_j).\qquad(15)$$

By applying Chebyshev or Legendre quadrature integration formula, (15) can be changed to the following form

$$C^T\Psi(x_i,y_j)-\sum_{l=0}^{N}\sum_{k=0}^{N}w_lw_kG(x_i,y_j,s_k,t_l,C^T\Psi(s_k,t_l))=f(x_i,y_j).\qquad(16)$$

This is a nonlinear system of equations that can be solved by Newton's iterative method to obtain the unknown vector C^T. We recall the obtained linearized system by Newton's method is ill-conditioned and the use of regularization methods is efficient. Also, we can apply some other iterative regularization methods for solving ill-conditioned nonlinear system [3].

5. Numerical results

In this section, the examples solved by collocation method based on Chebyshev polynomials and radial basis functions. The illustrate examples show efficiency of the mentioned methods, however complexity of mathematical operations is different for them. These example confirm RBF method is the same good as spectral methods or better than them. High speed convergence is technical characteristic of RBF method in higher dimensions. All of the computations have been done using the Maple 13 with 200 digits precision. In this study, our criterion of accuracy is the maximum of the absolute error in all of points of related intervals. In other words, we investigate the value of infinity norm of error functions.

Example 1. Consider the following Fredholm integral equation

$$u(x,t) - \int_{-1}^{1}\int_{-1}^{1}(zsinx + ty)u(y,z)dydz = x\cos t + \frac{4}{3}\sin x - (1 + \frac{4}{3}\sin(1))t,$$

with exact solution $u(x,t) = x\cos t - t$. By using (9) we obtain approximate solution. Also, we solve it again by using (12). The results are shown in Table 1.

Example 2. Consider the following Fredholm integral equation

$$u(x,t) - \int_{0}^{1}\int_{0}^{1}(xy + te^{z})u(y,z)dydz = xe^{-t} - \frac{1}{2}t - \frac{7}{12}x + \frac{1}{3}xe^{-1},$$

with exact solution $u(x,t) = xe^{-t} + t$. In the similar way, by using (9) and (12) we obtain approximate solution. The results are shown in Table 1.

Table 1: The first and second column(A_1 and B_1) are related to example 1 and the third and fourth column(A_2 and B_2) are related to example 2. A_1 and A_2 are the results by using of collocation method based on Chebyshev polynomials and B_1 and B_2 are the results based on radial basis functions.

N	A_1	B_1	A_2	B_2
2	3.02×10^{-2}	3.01×10^{-2}	6.23×10^{-3}	1.42×10^{-2}
4	1.21×10^{-5}	6.04×10^{-5}	1.98×10^{-5}	1.61×10^{-5}
6	3.49×10^{-7}	1.65×10^{-7}	1.23×10^{-7}	2.62×10^{-8}
8	9.95×10^{-10}	2.88×10^{-10}	1.18×10^{-7}	2.50×10^{-11}

6. conclusion

Analytical solution of the two dimensional integral equations are usually difficult. In many cases, it is required to approximate solutions. In this work, the two dimensional linear integral equations of the second kind is solved and compared by using Chebyshev polynomials and RBF method. The illustrative examples confirm the spectral convergence in both of conventional spectral method and RBF method. however, the best choice must be more effective in higher dimensions. There are a lot of sources that recommend to use radial basis functions in higher dimensions.

References

[1] A. Alipanah, S. Esmaeili, Numerical solution of the two-dimensional Fredholm integral equations by Gaussian radial basis function, J. Comput. Appl. Math. 2009, In Press.

[2] K. E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, 1997.

[3] B. Kaltenbacher, A. Neubauer, O. Scherzer, Iterative Regularization Methods for Nonlinear Ill-Posed Problems, Walter de Gruyter, 2008.

[4] J. P. Boyd, Chebyshev and Fourier Spectral Methods, DOVER Publication, Inc, 2000.

[5] M.D Buhmann, Radial Basis Functions, Cambridge University Press, 2003.

[6] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer-Verlag, New York, 1988.

[7] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, Cambridge University Press, 2004.

[8] W. Cheney, W. Light, A Course in Approximation Theory, William Allan, New York (1999).

[9] H. Guoqiang, K. Hayami, K. Sugihara, W. Jiong, Extrapolation method of iterated collocation solution for two-dimensional nonlinear Volterra integral equations, Appl. Math. Comput. 112 (2000), 49-61.

[10] H. Guoqiang, W. Jiong, Extrapolation of nystrom solution for two dimensional nonlinear Fredholm integral equations, J. Comput. Appl. Math. 134 (2001), 259-268.

[11] A. J. Jerri, Introduction to Integral Equations with Applications, John Wiley and Sons, INC, 1999.

[12] J. S. Hesthaven, S. Gottlieb, D. Gottlieb, Spectral Methods for Time-Dependent Problems, Cambridge University Press, 2007.

[13] K. Maleknejad, S. Sohrabi, B. Baranji, Application of 2D-BPFs to nonlinear integral equations, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010), 527-535.

[14] J. C. Mason, D. C. Handscomb, Chebyshev Polynomials, CRC Press LLC, 2003.

[15] W.R. Madych, Miscellaeous error bounds for multiquadratic and related interpolators, Comput. Math. Appl. 24 (1992), 121-138.

[16] C.A. Micchelli, Interpolation of scatter data: distance matrices and conditionally positive definite functions, Constr. Approx. 2 (1986), 11-22.

[17] A. Tari, M.Y. Rahimi, S. Shahmorad, F. Talati, Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method, J. Comput. Appl. Math. 228 (2009), 70-76.

[18] W. J. Xie, F. R. Lin, A fast numerical solution method for two dimensional Fredholm integral equations of the second kind, Appl. Math. Comput. 59 (2009), 1709-1719.

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