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Adaptive Stabilization and Control Stochastic Time-Varying Systems

Fakhreddin Abedi[1], **Malik Abu Hassan and Mohammed Suleiman**

Department of Mathematics and Institute for Mathematical Research
Universiti Putra Malaysia (UPM)
43400 UPM Serdang, Selangor, Malaysia

Abstract

We introduce the concept of an adaptive control Lyapunov function for the notion of globally asymptotically stable in probability of stochastic time-varying systems and use the stochastic version of Florchinger's control law [6] established in Abedi et al. [2] to design an adaptive controller. In this framework the problem of adaptive stabilization of a nonlinear stochastic system is reduced to the problem of nonadaptive stabilization of a modified system.

Mathematics Subject Classifications: 60H10; 93C10; 93D05; 93D15; 93D21; 93E15

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1 Introduction

The stabilization of various types of nonlinear stochastic differential system with Lyapunov theory has been studied for different notations of stochastic stability in recent years (see, for instance, Handel [7], Florchinger [6], Krstic and Deng [10], Deng et al. [5], Shaikhet [15], Abedi et al. [2], [3]).
Tsinias [19], Tsinias and Karafyllis [18], Karafyllis and Tsinias [8] have shown that for a class of triangular systems whose dynamics contain time-varying

[1]Corresponding author, e-mail: f_abedi1352@yahoo.com

unknown parameters, it is possible to find, by applying a backstepping design procedure, a smooth time-varying feedback controller in such a way that the equilibrium of the resulting closed-loop system is globally asymptotically stable, in general nonuniform with respect to initial values of time. Abedi et al. [1] have extended the well-known Artstein-Sontag theorem (see, for instance, Artstein [4], Sontag [16]) to the concepts of control Lyapunov function for the notion of nonuniform in time globally asymptotically stable in probability (GASP) for a class of stochastic time-varying systems (STVS).
The concept of an adaptive control Lyapunov function associated with deterministic nonlinear systems with unknown parameters has been introduced by Krstic and Kokotovic [11], and extended to nonlinear stochastic differential systems by Krstic and Deng [10], Deng et.al [5] and Abedi et al. [2].
The aim of this paper is to study the problem of GASP of STVS with an unknown constant parameters in the drift when both the drift and diffusion terms are affine in the control. We introduce the concept of an adaptive control Lyapunov function and use the stochastic version of Florchinger's control law [6] established in Abedi et al. [2] to design an adaptive stabilizer. In our framework the problem of adaptive stabilization of nonlinear stochastic differential systems is reduced to the problem of dynamic feedback stabilization, for all values of the unknown parameter of a modified system. The analysis used in this paper is closely related to that of Abedi et al. [2]. The main tools used in this paper are the stochastic Lyapunov theorem proved by Khasminskii [9] and the converse stability theorems of Kushner [12].
The paper is organized as follows. Section 2 introduces the class of affine in the control STVS and some basic definitions and results that we are dealing with in this paper. In section 3, we state and prove the main results of the paper on the feedback stabilization of the class of STVS with unknown constant parameters in the drift when both the drift and diffusion terms are affine in the control introduced in this section.

2 Stochastic Stability

The aim of this section is to recall some basic definitions and theorem about the Lyapunov function approach stochastic stability that we need in this paper. For a more detailed exposition of this subject refer to the books of Speyer and Chung [17] and Khasminiskii [9], and the paper of Abedi et al. [1].

Denote by (Ω, F, P) a probability space and w a standard R^m-valued Wiener process defined on this space. Consider the multi-input STVS in R^n

$$dx = f(t, x, d)dt + \sum_{k=1}^{m} h_k(t, x)dw \tag{1}$$

$$x \in R^n,\ d \in D,\ t \geq 0.$$

We assume that $D \subset R^e$ is a nonempty compact set and $f : R^+ \times R^n \times D \to R^n, h_k : R^+ \times R^n \to R^{n\times m}, 1 \leq k \leq m$, are mapping with $f(t, 0, d) = 0, h_k(t, 0) = 0$, for all $(t, d) \in R^+ \times D$ that satisfies the following hypotheses:
(i) The functions $f(t, x, d), h_k(t, x)$ are Borel measurable in t for all $(x, d) \in R^n \times D$.
(ii) The functions $f(t, x, d)$ is continuous in d for all $(t, x) \in R^+ \times R^n$.
(iii) The functions $f(t, x, d)$ and $h_k(t, x)$ are locally bounded and locally Lipschitz continuous in $x \in R^n$, uniformly in $d \in D$, in the sense that for every bounded interval $I \subset R^+$ and for every compact subset S of R^n, there exists a constant $K \geq 0$ such that

$$|f(t, x, d) - f(t, y, d)| + \sum_{k=1}^{m} |h_k(t, x) - h_k(t, y)| \leq K|x - y|,$$

$$\forall t \in I,\ (x, y) \in S \times S,\ d \in D,$$

and

$$|f(t, x, d)| + \sum_{k=1}^{m} |h_k(t, x)| \leq K(1 + |x|).$$

Notations. Throughout this paper we adopt the following notations:
• M_D denotes the set of all measurable functions from $R^+ = [0, +\infty)$ to D and functions $d \in M_D$ are time-varying parameters, where D is any given compact subset of R^e. For each $d \in M_D$, we denote by $x(t) = x(t_0, x_0, d)$ the solution of (1) at time t that corresponds to some input $d \in M_D$ initiated from x_0 at time t_0.
• K^+ denotes the class of positive nondecreasing C^∞ functions $\phi : R^+ \to (0, +\infty)$.

Definition 2.1 *A function* $\gamma : R^+ \to R^+$ *is*
$*$ *A* K*-function if it is continuous, strictly increasing and* $\gamma(0) = 0$,
$*$ *A* K_∞*-function if it is a* K*-function and also* $\gamma(r) \to \infty$ *as* $r \to \infty$ *and*
$*$ *A positive definite function if* $\gamma(r) > 0$ *for all* $r > 0$*, and* $\gamma(0) = 0$.

Definition 2.2 *The equilibrium* $x = 0$ *of the system (1) is*
• *globally stable in probability, if for every* $t_0 \geq 0, d \in M_D$ *and* $\varepsilon > 0$ *there exists a class* K*-function* $\gamma(.)$ *such that*

$$P\{|x(t)| < \gamma(|x_0|)\} \geq 1 - \varepsilon,$$

$$\forall x_0 \in R^n \backslash \{0\}.$$

• *GASP, if it is globally stable in probability and*

$$P\{\lim_{t\to\infty} |x(t)| = 0\} = 1,$$

$$\forall x_0 \in R^n.$$

Next, consider the multi-input STVS in R^n written in the Ito form

$$dx = \left(f(t,x,v) + \sum_{z=1}^{p} g_z(t,x)u^z \right) dt + \sum_{k=1}^{m} \left(h_k(t,x) + \sum_{z=1}^{p} q_{k,z}(t,x)u^z \right) dw \tag{2}$$

$$x \in R^n,\ v \in R^l,\ u \in R^p,\ t \geq 0,$$

where w is Wiener process and the dynamics $f(.), h_k(.), g_z : R^+ \times R^n \to R^{n\times p}, 1 \leq z \leq p$, and $q_{k,z} : R^+ \times R^n \to R^{n\times m\times p}, 1 \leq z \leq p, 1 \leq k \leq m$, are C^0 and locally Lipschitz with respect to (x,v) with $f(.,0,0) = 0$ and $h_k(.,0) = 0$. The main results of this section (Theorem 2.4) constitute the well-known Florchinger's control law [6] established in Abedi et al. [2] that guarantees existence of a feedback law $u = k(t,x)$ in such a way that the resulting closed-loop system

$$dx = \left(f(t,x,v) + \sum_{z=1}^{p} g_z(.)k(.)^z \right) dt + \sum_{k=1}^{m} \left(h_k(.) + \sum_{z=1}^{p} q_{k,z}(.)k(.)^z \right) dw \tag{3}$$

satisfies GASP with v as input.
Denoting by **D** the infinitesimal generator of the stochastic process solution of

the STVS (2), that is, $\mathbf{D}$ is the second-order differential operator defined for any function Φ in $C^2(R^+ \times R^n, R)$ by

$$\mathbf{D}\Phi(t,x) = \sum_{i=1}^{n} f^i(t,x,v)\frac{\partial \Phi(t,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} h_k^i(t,x)h_k^j(t,x)\frac{\partial^2 \Phi(t,x)}{\partial x_i \partial x_j}, \quad (4)$$

where $1 \leq i, j \leq n$.
For any $z \in (1, ..., p)$, denote by $\mathbf{D}_z$ the second order differential operator defined for any function Φ in $C^2(R^+ \times R^n, R)$ by

$$\mathbf{D}_z\Phi(x) = \sum_{i=1}^{n} g_z^i(t,x)\frac{\partial \Phi(x)}{\partial x_i} + \sum_{i,j=1}^{n}\sum_{k=1}^{m} h_k^i(t,x)q_{k,z}^j(t,z)\frac{\partial^2 \Phi(x)}{\partial x_i \partial x_j}, \quad (5)$$

and, for any $z, r \in (1, ..., p)$, denote by $\mathbf{D}_{z,r}$ the second order differential operator defined for any function Φ in $C^2(R^+ \times R^n, R)$ by

$$\mathbf{D}_{z,r}\Phi(t,x) = \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} q_{k,z}^i(t,x)q_{k,r}^j(t,x)\frac{\partial^2 \Phi(t,x)}{\partial x_i \partial x_j}.$$

Under a slight change of hypothesis (Definition 2.3 introduced by Abedi et al. [1], [2]) we obtain the notion of CLF as follows.

Definition 2.3 *Let $\gamma(t,x) : R^+ \times R^+ \to R^+$ be a positive definite function, which is C^0, locally Lipschitz in r, the STVS (2) admits a "* control Lyapunov function," *if there exists a C^2 positive definite function $\Phi : R^+ \times R^n \to R^+$, a class K_∞ functions a_1, a_2, and a positive definite function $\rho : R^+ \to R^+$ such that for all $(t,x,d) \in R^+ \times R^n \times D$ the following conditions hold:*

$$a_1(t,|x|) \leq \Phi(t,x) \leq a_2(t,|x|), \quad (6)$$

$$\boldsymbol{D}_z\Phi(t,x) = 0, \;\; \boldsymbol{D}_{z,r}\Phi(t,x) = 0, \;\; |v| \leq \gamma(t,|x|) \;\; \Rightarrow \quad (7)$$

$$\boldsymbol{D}\Phi(t,x) \leq -\rho(\Phi(t,x)) < 0.$$

In the following theorem, under a slight change of hypothesis (Theorem 3.3, or Theorem 3.2 introduced by Abedi et al. [2], [3], respectively) we exhibit GASP property for the resulting closed-loop system (3). The result of this theorem is a useful tool to design an adaptive controller in section 3.

Theorem 2.4 *Let Φ be a control Lyapunov function associated to the STVS (2), and for any $(t,x) \in R^+ \times R^n$, denote by $b(t,x)$ and $\Psi(t,x)$ the functions defined by*

$$b(t,x) = \boldsymbol{D}_z\Phi(t,x), \tag{8}$$

and

$$\Psi(t,x) = \max_{|v|\leq\gamma(t,|x|)} \boldsymbol{D}\Phi(t,x) + \rho(\Phi(t,x)), \tag{9}$$

then the feedback law

$$k(t,x) = -\frac{b(t,x)}{\Gamma(t,x)}, \tag{10}$$

where $b(.)$ is given by (8) and

$$\Gamma(t,x) = 1 + (\sup_{1\leq z,r\leq p} \boldsymbol{D}_{z,r}\Phi(t,x))^2, \tag{11}$$

renders the stochastic system (3) GASP with v as input.

Proof : Suppose Φ satisfies Definition 2.3, from (7) and definition (9) of $\Psi(t,x)$ it follows that

$$\mathbf{D}_z\Phi(t,x) = 0, \;\; \mathbf{D}_{z,r}\Phi(t,x) = 0 \;\; \Rightarrow \Psi(t,x) \leq 0. \tag{12}$$

Notice that from (10) and (11) the feedback law $k(t,x)$ is well defined for all (t,x), since the denominator in (10) is strictly positive for all $(t,x) \in R^+ \times R^n$, and is of class $C^0(R^+ \times R^n)$. Indeed, $\mathbf{D}_z\Phi(t,x) \geq 0$ for all $(t,x) \in R^+ \times R^n$, and suppose that $\mathbf{D}_z\Phi(t,x) = 0$ for certain $(t,x) \in R^+ \times R^n$. It then follows from (7) and definition (9) of $\Psi(.)$ that $\Psi(t,x) \leq 0$. Furthermore, according to regularity assumptions made for $\Phi(t,x), f(t,x,v), g_z(t,x), h_k(t,x)$, $\gamma(t,x)$ and $\rho(.)$, the map $k(t,x)$ as defined by (10) is C^0 on $R^+ \times R^n$ and locally Lipschitz with respect to $x \in R^n$, with $k(t,0) = 0$ for all $t \geq 0$.

Denoting by $\mathbf{D}_0$ the infinitesimal generator of the stochastic process solution of the resulting closed-loop system (3) we get

$$\begin{aligned}
\mathbf{D}_0\Phi(t,x) &= \max_{|v|\leq\gamma(t,|x|)} \mathbf{D}\Phi(t,x) + \sum_{z=1}^{p} \mathbf{D}_z\Phi(t,x)k(t,x) \\
&+ \sum_{z,r=1}^{p} \mathbf{D}_{z,r}\Phi(t,x)k(t,x)^2 \\
&= \max_{|v|\leq\gamma(t,|x|)} \mathbf{D}\Phi(t,x) - \frac{1}{\Gamma(t,x)} \sum_{z=1}^{p} (\mathbf{D}_z\Phi(t,x))^2 \\
&+ \frac{1}{\Gamma^2(t,x)} \sum_{z,r=1}^{p} \mathbf{D}_z\Phi(t,x)\mathbf{D}_r\Phi(t,x)\mathbf{D}_{z,r}\Phi(t,x).
\end{aligned}$$

Then, we have by taking into account (10) and (12) that

$$\mathbf{D}_0\Phi(t,x) \leq -\rho(\Phi(t,x)). \tag{13}$$

From (13) and $\Phi(t,x) \geq 0, \Phi_t = \Phi(t,x)$ is a supermartingale. The rest of the proof is straightforward consequence of (13) and the proof of Theorem 3.3 established in Abedi et al. [2]. $\square$

3 Adaptive Stabilization in Probability

Denote by $x(t) \in R^n$ the stochastic process solution of the STVS written in the sense of Ito:

$$\begin{aligned} dx = &\left(f(t,x,v) + \sum_{z=1}^{p} (M_z(t,x)\theta^z + g_z(t,x)u^z) \right) dt \\ &+ \sum_{k=1}^{m} \left(h_k(t,x) + \sum_{z=1}^{p} q_{k,z}(t,x)u^z \right) dw \\ &x \in R^n,\ v \in R^l,\ \theta \in R^q,\ u \in R^p,\ t \geq 0, \end{aligned} \tag{14}$$

where the dynamics $f(.), h_k(.), M_z(.), 1 \leq z \leq p, g_z(.)$ and $q_{k,z}(.)$ are C^0 and locally Lipschitz with respect to (x,v) with $f(.,0,0) = 0$, $h_k(.,0) = 0$, and $M_z(.,0) = 0$, and such that there exists a constant $K \geq 0$ such that

$$\sum_{z=1}^{p} |M_z(t,x)| \leq K(1+|x|),$$

for any $x \in R^n$.

The STVS (14) is said to be globally adaptively stabilizable in probability if there exists a function $\alpha(t,x,\hat{\theta})$ smooth on $(R^+ \times R^n \backslash \{0\}) \times R^q$ with $\alpha(.,0,\hat{\theta}) \equiv 0$, a smooth functional $\tau(t,x,\hat{\theta})$, and a positive definite symmetric matrix Γ in $M_{q\times q}(R)$ such that the dynamic control law

$$u = \alpha(t,x,\hat{\theta}), \tag{15}$$

$$\dot{\hat{\theta}} = \Gamma\tau(t,x,\hat{\theta}), \tag{16}$$

guarantees that the solution $(t,x,\hat{\theta})$ is globally bounded, and $x(t) \to 0$ as $t \to \infty$ for any value of the unknown parameter $\theta \in R^q$.

In the following we replace the problem of adaptive stabilization of (14) by a problem of nonadaptive stabilization of a modified system.

Definition 3.1 *A function Φ_a in $C^2(R^+ \times R^n \times R^q, R^+)$, is said to be an adaptive control Lyapunov function for the STVS (14) if there exists positive definite symmetric matrix Γ in $M_{q\times q}(R)$, functions $a_1, a_2 \in K_\infty$, and a C^0 positive definite function $\rho : R^+ \to R^+$, such that, for every $\theta \in R^q$, $\Phi_a(t, x, \theta)$ is a control Lyapunov function for the modified stochastic time-varying system (MSTVS)*

$$dx = \left(f(t,x,v) + \sum_{z=1}^{p} \left(M_z(t,x) \left(\theta^z + \Gamma \left(\frac{\partial \Phi_a}{\partial \theta}(.,\theta) \right)^T \right) + g_z(t,x)u^z \right) \right) dt + \sum_{k=1}^{m} \left(h_k(t,x) + \sum_{z=1}^{p} q_{k,z}(t,x)u^z \right) dw. \tag{17}$$

Definition 3.2 *The control Lyapunov function Φ_a defined above is said to satisfy the small control property if for each $\theta \in R^q$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $|x| \leq \delta$, then there is some u with $|u| \leq \varepsilon$ such that*

$$\boldsymbol{D}\Phi_a(.) + \sum_{z=1}^{p} \boldsymbol{D}_z\Phi_a(.)u^z + \sum_{z,r=1}^{p} \boldsymbol{D}_z\Phi_a(.)\boldsymbol{D}_{z,r}\Phi_a(.)(u^z)^2 + \rho(\Phi_a(.)) \leq 0.$$

We now show how to design an adaptive controller (15), (16) when an adaptive control Lyapunov function is known.

Theorem 3.3 *The following statements are equivalent:*
i. There exists a triple (α, Φ_a, Γ) such that $u = \alpha(t, x, \theta)$ globally asymptotically stabilizes in probability the MSTVS (17) at $x = 0$ for every θ in R^q with respect to the Lyapunov function $\Phi_a(t, x, \theta)$.
ii. There exists an adaptive control Lyapunov function $\Phi_a(t, x, \theta)$ for the STVS (14). Moreover, if an adaptive control Lyapunov function exists, then the STVS (14) is globally adaptively stabilizable in probability.

Proof : $(i \to ii)$ Since $u = \alpha(t, x, \theta)$ globally asymptotically stabilizes in probability the MSTVS (17), there exists a function Φ_a in $C^2(R^+ \times R^n \times R^q, R)$, positive definite in x for every $\theta \in R^q$, such that

$$\mathbf{D}_\theta\Phi_a(.) = \mathbf{D}_l\Phi_a(.) + \sum_{z=1}^{p} \mathbf{D}_z\Phi_a(.)u^z + \sum_{z,r=1}^{p} \mathbf{D}_z\Phi_a(.)\mathbf{D}_{z,r}\Phi_a(.)(u^z)^2 \tag{18}$$

$$\leq -\rho(\Phi_a(.)),$$

where $\mathbf{D}_\theta$ is the infinitesimal generator of the closed-loop system deduced from (17) when $u = \alpha(t, x, \theta)$, and

$$\mathbf{D}_l\Phi_a(.) = \sum_{i=1}^{n}\left(f^i(.) + \sum_{z=1}^{p} M_z^i(.)\left(\theta^z + \Gamma\left(\frac{\partial\Phi_a}{\partial\theta}(.)\right)^T\right)\right)\frac{\partial\Phi_a(.)}{\partial x_i} \tag{19}$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} h_k^i h_k^j \frac{\partial^2\Phi_a(.)}{\partial x_i \partial x_j},$$

$$\mathbf{D}_{z,r}\Phi_a(.) = \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} q_{k,z}^i(.) q_{k,r}^j(.)\frac{\partial^2\Phi_a(.)}{\partial x_i \partial x_j}, \tag{20}$$

$$\mathbf{D}_z\Phi_a(t, x, \theta) = \sum_{k=1}^{n} g_z^i(t, x)\frac{\partial\Phi_a(t, x, \theta)}{\partial x_i} + \sum_{i,j=1}^{n}\sum_{k=1}^{m} h_k^i q_{k,z}^j \frac{\partial^2\Phi_a(t, x)}{\partial x_i \partial x_j}, \tag{21}$$

for any $z, r \in (1, ..., p)$. The inequality (18) implies (7) with $\mathbf{D}_z\Phi_a(.) = 0$ and $\mathbf{D}_{z,r}\Phi_a(.) = 0$. Thus $\Phi_a(t, x, \theta)$ is a control Lyapunov function for the MSTVS (17) for every $\theta \in R^q$, and therefore it is an adaptive control Lyapunov function for STVS (14).

$(ii \to i)$ If $\Phi_a(t, x, \theta)$ is an adaptive control Lyapunov function for the STVS (14), then according to Definition 3.2 it is a control Lyapunov function for the MSTVS (17). Therefore, the control law

$$\alpha(t, x, \theta) = -\frac{\mathbf{D}_z\Phi_a(.)}{1 + (\sup_{1\leq z,r\leq p}\mathbf{D}_{z,r}\Phi_a(.))^2}, \tag{22}$$

where $\mathbf{D}_{z,r}, \mathbf{D}_z$, given by (20) and (21) respectively, is, according to Theorem 2.4, a smooth control law on $R^+ \times (R^n \backslash \{0\}) \times R^q$ which renders the equilibrium solution of the MSTVS (17) GASP for every $\theta \in R^q$. We note that the control law $u = \alpha(t, x, \theta)$ given by (22) will be continuous at $x = 0$ if and only if the adaptive control Lyapunov function Φ_a satisfies the small control property.

Assume that there exists an adaptive control Lyapunov function for the STVS (14). Since $(ii \to i)$, there exists a triple (α, Φ_a, Γ) and a continuous function Φ_a which is positive definite in x for every $\theta \in R^q$ such that (18) is satisfied, that is,

$$\mathbf{D}_\theta\Phi_a(t, x, \theta) \leq -\rho(\Phi_a(t, x, \theta)).$$

Let Φ be the Lyapunov function defined on $R^+ \times R^n \times R^q$ by

$$\Phi(t,x,\hat{\theta}) = \Phi_a(t,x,\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1} (\theta - \hat{\theta}). \quad (23)$$

Denoting by $\mathbf{D}$ the infinitesimal generator of the stochastic process $(x, \hat{\theta})$, the solution of (14)-(16), we find

$$\mathbf{D}\Phi(.) = \mathbf{D}_{\hat{\theta}}\Phi_a(.) + \frac{\partial \Phi_a}{\partial \hat{\theta}} \Gamma \tau(x,\hat{\theta}) - (\theta - \hat{\theta})^T \tau(x,\hat{\theta}) \quad (24)$$

$$-\frac{\partial \Phi_a}{\partial \hat{\theta}}(x,\hat{\theta})\Gamma \sum_{z=1}^{p} \left(\frac{\partial \Phi_a}{\partial x}(x,\hat{\theta}) M_z(.) \right)^T + (\theta - \hat{\theta})^T \sum_{z=1}^{p} \left(\frac{\partial \Phi_a}{\partial x}(x,\hat{\theta}) M_z(.) \right)^T .$$

Choosing

$$\tau(x,\hat{\theta}) = \sum_{z=1}^{p} \left(\frac{\partial \Phi_a}{\partial x}(x,\hat{\theta}) M_z(.) \right)^T , \quad (25)$$

we get

$$\mathbf{D}\Phi(x,\hat{\theta}) \leq -\rho(\Phi_a(t,x,\theta)), \quad (26)$$

for every $\theta \in R^q$.
Therefore, by means of martingale theory argument (see Rogers and Williams [14]) and the stochastic version of La Salle's invariance theorem (see Kushner [13]) as shown in the proof of Theorem 2.4, the equilibrium solution of the STVS (14) is globally adaptively stabilizable in probability. □

The control law $u = \alpha(t,x,\theta)$ given by (22) renders the MSTVS (17) GASP but it may not be a stabilizer for the original STVS (14). However, as shown in the proof of Theorem 3.3, the feedback law $u = \alpha(t,x,\hat{\theta})$ given by (22) and the update law $\dot{\hat{\theta}} = \Gamma\tau(t,x,\hat{\theta})$ with (25) is an adaptive stabilizing feedback law for the STVS (14). We will now show that the quadratic form of the Lyapunov function (23) is both necessary and sufficient for the existence of an adaptive control Lyapunov function.

Definition 3.4 *The STVS (14) is globally adaptively quadratically stabilizable in probability if it is globally adaptively stabilizable in probability and there exists a function Φ_a in $C^2(R^+ \times R^n \times R^q, R)$, and a C^0 positive definite function $\rho : R^+ \to R^+$, such that, for every $\theta \in R^q$,*

$$\boldsymbol{D}\Phi(t,x,\hat{\theta}) \leq -\rho(\Phi_a(t,x,\hat{\theta})),$$

where Φ is the Lyapunov function given by (23).

Corollary 3.5 *The STVS (14) is globally adaptively quadratically stabilizable in probability if and only if there exists an adaptive control Lyapunov function.*

Proof : The necessary part of the results is contained in the proof of Theorem 3.3 where the Lyapunov function $\Phi(t, x, \hat{\theta})$ is in the form (23).
For proof of the sufficient part, we assume, that the STVS (14) is globally adaptively quadratically stabilizable in probability and we first prove that $\tau(t, x, \hat{\theta})$ must be given by (25). Equality (24) can be rewritten as

$$\mathbf{D}\Phi(x,\hat{\theta}) = \mathbf{D}_{\hat{\theta}}\Phi_a + \frac{\partial \Phi_a}{\partial \hat{\theta}}\Gamma\tau(x,\hat{\theta}) - \frac{\partial \Phi_a}{\partial \hat{\theta}}(x,\hat{\theta})\Gamma\sum_{z=1}^{p}\left(\frac{\partial \Phi_a}{\partial x}(x,\hat{\theta})M_z(.)\right)^T \quad (27)$$

$$-\hat{\theta}^T\left(\sum_{z=1}^{p}\left(\frac{\partial \Phi_a}{\partial x}(x,\hat{\theta})M_z(.)\right)^T - \tau(x,\hat{\theta})\right)$$

$$+\theta^T\left(\sum_{z=1}^{p}\left(\frac{\partial \Phi_a}{\partial x}(x,\hat{\theta})M_z(.)\right)^T - \tau(x,\hat{\theta})\right).$$

Then, for the STVS (14) to be globally adaptively quadratically stabilizable in probability, this expression has to be nonpositive to satisfy (26). Therefore, since the right-hand side of equality (27) is affine in θ, it is nonpositive for every $x \in R^n$ and $\theta, \hat{\theta} \in R^q$ only if the last term is zero; that is, if

$$\tau(t,x,\hat{\theta}) = \sum_{z=1}^{p}\left(\frac{\partial \Phi_a(.)}{\partial x}M_z(.)\right)^T.$$

Furthermore, in this case, it can be easily deduced from $(i \rightarrow ii)$ in Theorem 3.3 that $\Phi_a(t, x, \theta)$ is an adaptive control Lyapunov function for the STVS (14). □

4 Conclusions

In this paper, we have introduced the concept of an adaptive control Lyapunov function for the notion of GASP of STVS with an unknown constant parameter in the drift and we used the stochastic version of Florchinger's control law [6] established in Abedi et al. [2] to design an adaptive controller. The adaptive control Lyapunov function framework reduces the problem of adaptive stabilization to the problem of nonadaptive stabilization of a modified system.

The problem of adaptive stabilization in probability is difficult because the functional Φ_a used in our framework, which modifies the stochastic differential system (17), has to be its Lyapunov function.

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