Applied Mathematical Sciences, Vol. 5, 2011, no. 24, 1171 - 1176

The Frenet Frame and Darboux Vector of the Dual Curve on the One-Parameter Dual Spherical Motion

Nemat Abazari

Department of Mathematics
Islamic Azad University-Ardebil Branch
P.O. Box 5614633167, Ardebil, Iran
Nemat.Abazari@science.ankara.edu.tr

Abstract. In this paper, the tangent, binormal, normal and unit Darboux vectors of the dual curve on the one–parameter dual spherical motion are obtained with respect to a unit dual orthogonal frame for dual 3–space D^3.

Keywords: Dual vector, Dual spherical motion, Dual unit Frenet frame, Dual Darboux vector, Pfaffian vector

1. Introduction

The analysis of spatial motions in differential geometry [1,2,7,8] and in the kinematics of the spatial mechanisms [9], [10], [3] the use of dual vectors, dual quaternions and dual matrix algebra over the ring of dual numbers is a very direct method. Important properties of a real vector analysis of real matrix algebra are valid for the dual vectors and dual matrices. The principle part of this method is based on work by E. STUDY [9]. The essential idea is to replace points by straight lines as a fundamental building blocks of geometric begins. The set of oriented lines in Euclidean three dimensional space $\mathbb{R}^3$ is in one-to-one correspondence with the points of a unit dual sphere in the dual space D^3 of triples of dual numbers.

In this paper, we calculate the Frenet frame and Darboux vector of the dual curve (X) for the dual-spherical motion K/K' on D-modul, corresponds to one-parameter motion H/H', where H' and H denote the fixed and moving line-spaces, respectively.

2. Basic concept

Is a and $a^\star$ are real numbers and $\epsilon^2 = 0$, the combination $A = a + \epsilon a^\star$ is called a dual number, where ϵ is a dual unit. The set of all dual numbers form a commutative ring over the real number field and it is denote by D.Then the set

$$D^3 = \{\tilde{a} = (A_1, A_2, A_2) | A_i \in D,\ 1 \leq i \leq 3\}$$

is a module over the ring D which is called a D–module or dual space. The elements of D^3 are called dual vectors. Thus a dual vector $\tilde{a}$ can be written $\tilde{a} = a + \epsilon a^\star$ where a and $a^\star$ are real vectors at $\mathbb{R}^3$. The product of dual vectors $\tilde{a}$ and $\tilde{b}$ is defined by $\langle \tilde{a}, \tilde{b} \rangle = \langle a, b \rangle + \epsilon(\langle a, b^\star \rangle + \langle a^\star, b \rangle)$ with $\tilde{a} = a + \epsilon a^\star$ and $\tilde{b} = b + \epsilon b^\star$. The unit spheres is

$$S_1^2 = \{\tilde{a} = a + \epsilon a^\star \in D_1^3 |\ \langle \tilde{a}, \tilde{a} \rangle = 1; a, a^\star \in \mathbb{R}^3\}.$$

3. The dual spherical motion

Let H' and H denote the fixed and moving line spaces, respectively. According to the E-Study Theorem, unit dual spheres K' and K centered M correspond to these spaces on D-module, respectively. Also, K/K' dual spherical motion corresponds to H/H' one-parameter motion. Let us take into consideration a $\overrightarrow{X}$ line on H. That is to say, let we consider the constant point X of the unit dual curve K. On H motion, the $\overrightarrow{X}$ line traces a ruled surface (X) called orbit surface on H'. The variation of the point X up on K' i.e. the variation of the line $\overrightarrow{X}$ up on H' is $d_f\overrightarrow{X} = \overrightarrow{\Psi} \wedge \overrightarrow{X}$ where the vector $\overrightarrow{\Psi} = \vec{\psi} + \varepsilon\overrightarrow{\psi^*} = (\Psi_1, \Psi_2, \Psi_3)$ is called as the instantaneous-Pfaffian vector of the motion K/K'.

The ruled surface (X) is denoted by $\overrightarrow{X} = \overrightarrow{X}(t) = \overrightarrow{x}(t) + \varepsilon\overrightarrow{x^*}(t)$ where $\overrightarrow{X} = \overrightarrow{X}(t)$ is the unit dual vectorial function parameterized by $t \in \mathbb{R}$. The dual curve (X) is the dual spherical formation of the ruled surface. $d\Phi^2 = \langle d\overrightarrow{X}, d\overrightarrow{X} \rangle = \langle d\overrightarrow{x}, d\overrightarrow{x} \rangle + 2\varepsilon\langle d\overrightarrow{x}, d\overrightarrow{x^*} \rangle$ is valid for the $d\Phi = d\varphi + \varepsilon d\varphi^*$ dual arch element of the dual curve $\overrightarrow{X} = \overrightarrow{X}(t)$ on unit dual sphere. The pitch of a closed ruled surface was studied by Hacisalihoglu in [5,6] and the theorem in areas including Holditch's, with its analogue in three dimensions was audited by Elliott in [4].

Remark . In this paper we suppose that $\{\vec{R}_1, \vec{R}_2, \vec{R}_3\}$ be a unit dual orthogonal frame for dual 3–space D^3, such that $\vec{R}_i = \vec{r}_i + \epsilon\vec{r}_i^\star$ for $i = 1, 2, 3$. Also, $\vec{R}_3 = \vec{R}_1 \wedge \vec{R}_2,\ \vec{R}_2 = \vec{R}_3 \wedge \vec{R}_1,\ \vec{R}_1 = \vec{R}_2 \wedge \vec{R}_3$, and $\langle \vec{R}_3, \vec{R}_1 \rangle = 0,\ \langle \vec{R}_2, \vec{R}_3 \rangle = 0,\ \langle \vec{R}_1, \vec{R}_2 \rangle = 0$, and $\langle \vec{R}_i, \vec{R}_i \rangle = 1, \quad$ for $i = 1, 2, 3$.

Definition . Let K be the moving unit dual sphere. On condition that the pitch of the motion p is non-vanishing, a new coordinate system is carried with the $\overrightarrow{P} = \overrightarrow{R_3}$ system privately. This system is called as canonical coordinate system. For this case, $\overrightarrow{\Psi} = \Psi_3\overrightarrow{R_3} = \Psi_3\overrightarrow{P}$ is the instantaneous Pfaffian vector.

The declaration of the variation of the point (X) according to canonical coordinate system on the one-parameter motion K/K' is:

$$d_f\overrightarrow{X} = \overrightarrow{\Psi} \wedge \overrightarrow{X} = \Psi_3\left(\overrightarrow{P} \wedge \overrightarrow{X}\right) = \Psi_3\left(\overrightarrow{R_3} \wedge \overrightarrow{X}\right) = \Psi_3\left(X_1\overrightarrow{R_2} - X_2\overrightarrow{R_1}\right)$$

where $\overrightarrow{\Psi} = \Psi_3\overrightarrow{R_3}$ and $\overrightarrow{X} = X_1\overrightarrow{R_1} + X_2\overrightarrow{R_2} + X_3\overrightarrow{R_3}$.

4. The Frenet frame and Darboux vector of the dual curve

On the one-parameter dual spherical motion, the constant point $X \in K$ constructs a dual curve on K'. The tangent, binormal and normal of the dual curve at the point (X) are:

$$\overrightarrow{T} = \frac{d_f\overrightarrow{X}}{\left\|d_f\overrightarrow{X}\right\|} = \frac{\overrightarrow{\Psi} \wedge \overrightarrow{X}}{\left\|\overrightarrow{\Psi} \wedge \overrightarrow{X}\right\|}, \quad \overrightarrow{B} = \frac{d_f\overrightarrow{X} \wedge d_f^2\overrightarrow{X}}{\left\|d_f\overrightarrow{X} \wedge d_f^2\overrightarrow{X}\right\|}, \quad \overrightarrow{N} = \overrightarrow{B} \wedge \overrightarrow{T}$$

respectively.

Theorm . On the one-parameter motion K/K', the tangent, binormal and normal vectors at the point (X) are:

$$\overrightarrow{T} = \frac{1}{\sqrt{1-X_3^2}}\left(-X_2\overrightarrow{R_1} + X_1\overrightarrow{R_2}\right), \quad \overrightarrow{N} = \frac{-1}{\sqrt{1-X_3^2}}\left(X_1\overrightarrow{R_1} + X_2\overrightarrow{R_2}\right), \quad \overrightarrow{B} = \overrightarrow{R_3}$$

respectively, where $X = (X_1, X_2, X_3)$.

proof . Suppose that $R = \begin{pmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vec{R}_3 \end{pmatrix}$, where $\{\vec{R}_1, \vec{R}_2, \vec{R}_3\}$ are a unit dual orthogonal frame for 3–space D^3. Since, X is on unit dual sphere K, then

$$\vec{X} = X_1\vec{R}_1 + X_2\vec{R}_2 + X_3\vec{R}_3 = X^T R, \qquad \|\vec{X}\|^2 = X_1^2 + X_2^2 + X_3^2 = \mathbf{1} = (1,0)$$

where $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, and the $\vec{X}$ is the dual vector corresponds of X. The displacement with respect to K and K', the dual moving and fixed sphere, respectively, $dR = \Omega R$ and $d'R = \Omega' R$, where

$$\Omega = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}, \quad \Omega' = \begin{pmatrix} 0 & \Omega_3' & -\Omega_2' \\ -\Omega_3' & 0 & \Omega_1' \\ \Omega_2' & -\Omega_1' & 0 \end{pmatrix}$$

then, the displacement of $\vec{X}$ with respect to K and K', will be respectively,

$$d\vec{X} = dX^T R + X^T dR = dX^T R + X^T\Omega R = (dX^T + X^T\Omega)R, \tag{4.4.1}$$

and

$$d'\vec{X} = d'X^T R + X^T d'R = d'X^T R + X^T \Omega' R = (d'X^T + X^T \Omega')R, \tag{4.4.2}$$

since, Ω and Ω' are antisymmetric matrixes, then $\Omega^T = -\Omega$ and $\Omega'^T = -\Omega'$, For any fixed vector $\vec{X}$, we get $d\vec{X} = 0$ and $d'\vec{X} = 0$, therefore from Eqs. (4.4.1) and (4.4.2), we get

$$dX^T = -X^T\Omega = X^T\Omega^T, \tag{4.4.3}$$

also,

$$dX^T = -X^T\Omega' = X^T\Omega'^T, \tag{4.4.4}$$

Now, suppose that X is fixed in K and let us calculate its velocity $d_f X$ with respect to K', then we substitute (4.4.3) in (4.4.4), and obtain

$$d_f\vec{X} = d'\vec{X} - d\vec{X} = (X^T\Omega' - X^T\Omega)R = X^T(\Omega' - \Omega)R,$$

If we define a new dual vector whose components in the relative system are $\Psi_i = \Omega'_i - \Omega_i$, where $i = 1, 2, 3$, and $\vec{\Psi} = (\Psi_1, \Psi_2, \Psi_3) = \Psi_1\vec{R}_1 + \Psi_2\vec{R}_2 + \Psi_3\vec{R}_3$, then we get $d_f\vec{X} = \vec{\Psi} \wedge \vec{X}$, where $\vec{\Psi}$ is the Pfaffian vector correspond to the dual spherical motion K/K'. To calculate the acceleration $J = d_f^2\vec{X}$ we have

$$J = d_f^2\vec{X} = \vec{\Psi} \wedge (\vec{\Psi} \wedge \vec{X}) + \vec{\Psi} \wedge \vec{X} = -\Psi^2\vec{X} + < \vec{\Psi}, \vec{X} > \vec{\Psi} + \vec{\Psi} \wedge \vec{X} \tag{4.4.5}$$

by using the matrix model, the relations (3) and (4) yields

$$d_f X = MX$$

and

$$J = d_f^2 X = (M^2 + \dot{M})X$$

where

$$M = \begin{pmatrix} 0 & -\Psi_3 & \Psi_2 \\ \Psi_3 & 0 & -\Psi_1 \\ -\Psi_2 & \Psi_1 & 0 \end{pmatrix}, \ \dot{M} = \begin{pmatrix} 0 & -\dot{\Psi}_3 & \dot{\Psi}_2 \\ \dot{\Psi}_3 & 0 & -\dot{\Psi}_1 \\ -\dot{\Psi}_2 & \dot{\Psi}_1 & 0 \end{pmatrix}$$

then we obtain

$$M^2 = \begin{pmatrix} -\Psi_3^2 - \Psi_2^2 & \Psi_1\Psi_2 & \Psi_1\Psi_3 \\ \Psi_1\Psi_2 & -\Psi_3^2 - \Psi_1^2 & \Psi_2\Psi_3 \\ \Psi_1\Psi_3 & \Psi_2\Psi_3 & -\Psi_2^2 - \Psi_1^2 \end{pmatrix} = \begin{pmatrix} -\Psi^2 - \Psi_1^2 & \Psi_1\Psi_2 & \Psi_1\Psi_3 \\ \Psi_1\Psi_2 & \Psi^2 + \Psi_2^2 & \Psi_2\Psi_3 \\ \Psi_1\Psi_3 & \Psi_2\Psi_3 & -\Psi^2 - \Psi_3^2 \end{pmatrix}$$

If $\vec{\Psi} = \Psi_3\vec{R}_3$ then $\Psi_1 = \Psi_2 = 0$ therefore

$$M = \begin{pmatrix} 0 & -\Psi_3 & 0 \\ \Psi_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$d_f\vec{X} = MX = \begin{pmatrix} -\Psi_3 X_2 \\ \Psi_3 X_1 \\ 0 \end{pmatrix} = \Psi_3 X_2 \vec{R_1} - \Psi_3 X_1 \vec{R_2} = \Psi_3 (X_2 \vec{R_1} - X_1 \vec{R_2}) \vec{T}$$

hence

$$\frac{d_f\vec{X}}{\| d_f\vec{X} \|} = \frac{\Psi_3 (X_2 \vec{R_1} - X_1 \vec{R_2})}{\sqrt{\Psi_3^2 (X_2^2 + X_1^2}} = \frac{1}{\sqrt{1 - X_3^2}} (X_2 \vec{R_1} - X_1 \vec{R_2}).$$

By using the same manner, it is easy to see that $\vec{B} = \vec{R}_3$, therefore

$$\begin{aligned} \vec{N} = \vec{B} \wedge \vec{T} = \vec{R}_3 \wedge \frac{1}{\sqrt{1 - X_3^2}} (X_2 \vec{R_1} - X_1 \vec{R_2}) \\ = \frac{1}{\sqrt{1 - X_3^2}} (X_2 \vec{R_3} \wedge \vec{R_1} - X_1 \vec{R_3} \wedge \vec{R_2}) \\ = \frac{1}{\sqrt{1 - X_3^2}} (X_2 \vec{R_2} + X_1 \vec{R_1}). \end{aligned}$$

From the previous Theorem, the following corollary can be obtained.

Corollary . On the one-parameter motion K/K', the unit Darboux vector is

$$\overrightarrow{D_0} = \frac{\tau \overrightarrow{T} + k \overrightarrow{B}}{\sqrt{\tau^2 + k^2}} = \frac{1}{\sqrt{\tau^2 + k^2}} \left[\frac{\tau}{\sqrt{1 - X_3^2}} \left(-X_2 \overrightarrow{R_1} + X_1 \overrightarrow{R_2} \right) + k \overrightarrow{R_3} \right]$$

where k and τ are the first and second dual curvatures respectively for $\vec{X} = (X_1, X_2, X_3)$.

5. Acknowledgment

The authors would like to thank Prof. Dr. H. H. Hacisalihoglu and Prof. Reza. Abazari for there careful reading of the first draft and many helpful suggestions.

References

[1] BLASCHKE W., *Zur Bewegungsgeometrie auf der Kugel, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Math.-Natur. Wiss. Klasse. 2*, Abhandlung 3, 31-37; (1948).

[2] BLASCHKE W. and MULLER H. R., *Ebene Kinematik*, R. Oldenbourg, 114-143; Munchen (1956).

[3] DIMENTBERG F.M., *The screw calculus and its applications in mechanics*, U.S. Technical Translation Number FTD-HT-23-1632-67, (1968).

[4] ELLIOTT E. B., *A theorem in areas including Holditch's, with its analogue in three dimensions*, The Messenger of Mathematics 7, 150-156 (1878); 10, 47-57 (1881).

[5] HACISALIHOGLU H. H., *On the pitch of a closed ruled surface*, Mechanism and Machine Theory, 7, 291-305, (1970).
[6] HACISALIHOGLU H. H., *On closed spherical motions*, Q. Appl. Math. 29, 269-279 (1971).
[7] KEMPE A. B., *Note on Mr. Leudesdorf's theorem in kinematics*, The Messenger of Mathematics 7, 165-167 (1878).
[8] LEUDESDORF C., *Theorem in kinematics*, The Messenger of Mathematics 7, 125-127; and 8, 11-12, (1877).
[9] STUDY E., *Geometrie der Dynamen*, B. G. Teubner, Leipzig (1903).
[10] YANG A.T., *Application of quaternion algebra and dual numbers to the analysis of spatil mechanisms*, Doctoral Dissertion, Columbia Univ., (1963).

Received: October, 2010