Applied Mathematical Sciences, Vol. 5, 2011, no. 1, 35 - 51

An $M^{[X]}/G/1$ Queue with Bernoulli Schedule, General Vacation Times, Random Breakdowns, General Delay Times and General Repair Times

Rehab F. Khalaf [1], Kailash C. Madan [2] and Cormac A. Lukas [3]

[1] School of Information Systems Computing and Mathematics
Brunel University, United Kingdom
Rehab.Khalaf@brunel.ac.uk, rihabfadil@gmail.com

[2] College of Mathematical Sciences and IT, Ahlia University
Kingdom of Bahrain,
kailash@ahliauniversity.edu.bh

[3] School of Information Systems Computing and Mathematics
Brunel University, United Kingdom
cormac.lucas@brunel.ac.uk

Abstract

We study an $M^{[X]}/G/1$ queuing system with Bernoulli schedule server vacations and random system breakdowns. The main new assumption in this paper is that the repair process does not start immediately after a breakdown and there is a 'delay time' waiting for repairs to start. We assume that the service times, vacation times, delay times and repair times all have a general distribution while the 'time to breakdown' is exponentially distributed. The steady state solutions have been found by using Supplementary variables technique.

1. Introduction

Queuing systems with server vacations and/or random system breakdowns have been studied by numerous researchers including Avi and Naor (1963), Kulkarni and Choi (1990), Takine and Sengupta (1997), Wang and Li (2001), Madan, Abu-Dayyeh and Gharaibeh (2003b) and Maraghi, Madan, and Dowman (2009a), to

mentions few. In most of the papers including the ones mentioned above the authors assume that whenever the system breaks down the repair process starts immediately. However, this is not the case in many real life situations. It is usually a common phenomenon that as a result of a sudden breakdown, the system has to wait for repair to start. We term this waiting time as 'delay time' and the focus of this paper is to study the effect of this 'delay time' on the system.

Most of the recent studies have been devoted to batch arrival vacation models under different vacation policies because of its interdisciplinary character. In this paper, we consider a batch arrival queueing system $M^{[X]}/G/1$, in which we assume that after every service completion, the server has the option to leave for a vacation of random length with probability p or to continue staying in the system with probability $1-p$. This is termed as 'modified Bernoulli schedule vacation' considered by Madan, Abu-Dayyeh and Gharaibeh (2003b) and Maraghi,

Madan, and Dowman (2009). This vacation period has been assumed to have general distribution. On the termination of a vacation, the server immediately joins the system to serve the waiting customers. In addition to optional server vacations, the system suffers random breakdowns from time to time. The 'delay time' before the repair process as well as the 'repair time' after the delay are both assumed to have a general distribution. It is further assumed that the system starts working immediately after its repairs are complete.

We assume that the customers arrive to the service station in batches of variable size, but are served one by one. We assume that the service times, vacation times, repair times and delay times each have a general distribution while the 'time to breakdown' is exponentially distributed.
The rest of this paper is organized as follows: the assumptions underlying the mathematical model are given in section 3.2. In section 3.3 we introduce the definitions and notation of this chapter, while in section 3.4 all the steady state equations governing the mathematical system formulated. In section 3.5 Supplementary variables technique has been used to obtain steady state results in explicit and closed form in terms of the probability generating functions for the number of customers in the queue, the average number of customers, and the average waiting time in the queue. Tow particular cases are given in section 3.6 to show the relations between this work and previous works done by other searchers. In section 3.7 the mean number of customers in the queue under the steady state has been found. Numerical illustration is given in section 3.8.

2. Mathematical Model

Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i \Delta t\,(i = 1,2,3,...)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+\Delta t)$, where

$0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches. The customers are provided service one by one on a 'first come- first served basis'. There is a single server and the service time follows a general (arbitrary) distribution with distribution function $G(s)$ and density function $g(s)$. Let $\mu(x)\Delta x$ be the conditional probability density of service completion during the interval, $(x, x+\Delta x]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{g(x)}{1-G(x)} \tag{1}$$

and, therefore

On completion of service, the server may take a vacation of random length with

$$g(s) = \mu(s)e^{-\int_0^s \mu(x)dx} \tag{2}$$

probability p, or may stay in the system providing service with probability $1-p$, where $0 \le p \le 1$. The server's vacation time follows a general (arbitrary) distribution with distribution function $B(v)$ and density function $b(v)$. Let $\beta(x)\Delta x$ be the conditional probability of a completion of a vacation during the interval $(x, x+\Delta x]$ given that the elapsed vacation time is x, so that

$$\beta(x) = \frac{b(x)}{1-B(x)} \tag{3}$$

and, therefore

$$b(v) = \beta(v)e^{-\int_0^v \beta(x)dx} \tag{4}$$

The system may breakdown at random, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue. Once the system breakdown, its repairs do not start immediately and there is a delay time. The delay times follows a general (arbitrary) distribution with distribution function $W(x)$ and density function $w(x)$. Let $\varphi(x)\Delta x$ be the conditional probability of a completion of a delay during the interval $(x, x+\Delta x]$ given that the elapsed delay time is x, so that

$$\varphi(x) = \frac{w(x)}{1-W(x)} \tag{5}$$

and, therefore

$$w(x) = \varphi(x)e^{-\int_0^x \varphi(t)dt} \tag{6}$$

The duration of repair follows a general (arbitrary) distribution with distribution function $H(r)$ and density function $h(r)$. Let $\gamma(x)\Delta x$ be the conditional probability of a completion of a repair during the interval $(x, x+\Delta x]$ given that the elapsed repair time is *x*, so that

$$\gamma(x) = \frac{h(x)}{1-H(x)} \tag{7}$$

and, therefore

$$h(r) = \gamma(r) e^{-\int_0^r \gamma(x)dx} \tag{8}$$

All stochastic processes involved in the system are independent of each other.

3. Definitions and notations

We let $P_n(t,x)$ to be the probability that at time *t*, there are $n \geq 0$ customers in the queue excluding one customer in service and the elapsed service time of this customer is *x*. Accordingly, $P_n(t) = \int_0^\infty P_n(t,x)dx$ denotes the probability that there are $n \geq 1$ customers in the queue excluding one customer in service irrespective of the value of *x*. Next we let $V_n(t,x)$ to be the probability that at time *t*, there are $n \geq 0$ customers in the queue and the server is on vacation with elapsed vacation time *x*. Accordingly, $V_n(t) = \int_0^\infty V_n(t,x)dx$ denotes the probability that at time *t*, there are $n \geq 0$ customers in the queue and the server is on vacation irrespective of the value of *x*. More we consider $Q(t)$ is the probability that at time *t*, there are no customers in the system and the server is idle but available in the system. $D_n(t,x)$ is the probability that at time *t*, there are *n* ($n \geq 0$) customers in the queue, and the server is inactive due to system breakdown and waiting for repairs to start with elapsed delay time *x*. Accordingly, $D_n(t) = \int_0^\infty D_n(t,x)dx$ denotes the probability that at time *t*, there are $n \geq 0$ customers in the queue and the server is waiting for repairs to start irrespective of the value of *x*. Finally, let $R_n(t,x)$ be the probability that at time *t*, there are *n* ($n \geq 0$) customers in the queue, and the server is under repair with elapsed repair time *x*. Accordingly, $R_n(t) = \int_0^\infty R_n(t,x)dx$ denotes the probability that at time *t*, there are $n \geq 0$ customers in the queue and the server under repairs irrespective of the value of *x*. Assuming that the steady state exists, we let

$$\lim_{t\to\infty} P_n(t,x) = P_n(x), \quad \lim_{t\to\infty} P_n(t) = \lim_{t\to\infty}\int_0^\infty P_n(t,x)dx = P_n$$

$$\lim_{t\to\infty} V_n(t,x) = V_n(x), \quad \lim_{t\to\infty} V_n(t) = \lim_{t\to\infty}\int_0^\infty V_n(t,x)dx = V_n$$

$$\lim_{t\to\infty} R_n(t,x) = R_n(x), \quad \lim_{t\to\infty} R_n(t) = \lim_{t\to\infty}\int_0^\infty R_n(t,x)dx = R_n . \tag{9}$$

$$\lim_{t\to\infty} D_n(t,x) = D_n(x), \quad \lim_{t\to\infty} D_n(t) = \lim_{t\to\infty}\int_0^\infty D_n(t,x)dx = D_n \ , \lim_{t\to\infty} Q(t) = Q$$

$$\lim_{t\to\infty}\frac{dP_n(t)}{dt} = 0 \ , \ \lim_{t\to\infty}\frac{dV_n(t)}{dt} = 0 \ , \ \lim_{t\to\infty}\frac{dR_n(t)}{dt} = 0, \ \lim_{t\to\infty}\frac{dD_n(t)}{dt} = 0, \quad n \geq 0$$

4. Equations Governing the System

According to the mathematical model mentioned above, the system has the following set of differential-difference equations

$$\frac{\partial}{\partial x}P_n(x) = -(\lambda + \mu(x) + \alpha)P_n(x) + \lambda\sum_{i=1}^{n-1} c_i P_{n-i}(x) \tag{10}$$

$$\frac{\partial}{\partial x}P_0(x) = -(\lambda + \mu(x) + \alpha)P_0(x) \tag{11}$$

$$\frac{\partial}{\partial x}V_n(x) = -(\lambda + \beta(x))V_n(x) + \lambda\sum_{i=1}^{n} c_i V_{n-i}(x), \qquad n \geq 1 \tag{12}$$

$$\frac{\partial}{\partial x}V_0(x) = -(\lambda + \beta(x))V_0(x) \tag{13}$$

$$\frac{\partial}{\partial x}D_n(x) + (\lambda + \varphi(x))D_n(x) = \lambda\sum_{i=1}^{n} c_i D_{n-i}(x) \quad n \geq 1 \tag{14}$$

$$\frac{\partial}{\partial x}D_0(x) = 0 \tag{15}$$

$$\lambda Q = \int_0^\infty R_0(x)\gamma(x)dx + \int_0^\infty V_0(x)\beta(x)dx + (1-p)\int_0^\infty P_0(x)\mu(x)dx \tag{16}$$

$$\frac{\partial}{\partial x}R_n(x) + (\lambda + \gamma(x))R_n(x) = \lambda\sum_{i=1}^{n} c_i R_{n-i}(x) \quad n \geq 1 \tag{17}$$

$$\frac{\partial}{\partial x}R_0(x) = -(\lambda + \gamma(x))R_0(x) \tag{18}$$

The following boundary conditions will be used to solve the above equations

$$\begin{aligned} P_n(0) &= (1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx + \int_0^\infty V_{n+1}(x)\beta(x)dx \\ &+ \int_0^\infty R_{n+1}(x)\gamma(x)dx + \lambda c_{n+1}Q \qquad n \geq 0 \end{aligned} \tag{19}$$

$$V_n(0) = p\int_0^\infty P_n(x)\mu(x)dx, \qquad n \geq 0 \tag{20}$$

$$\begin{aligned} D_n(0) &= \alpha\int_0^\infty P_{n-1}(x)dx \qquad n \geq 1 \\ D_n(0) &= \alpha P_{n-1} \qquad\qquad n \geq 1 \end{aligned} \tag{21}$$

$$D_0(0) = 0 \tag{22}$$

$$R_n(0) = \int_0^\infty D_n(x)\varphi(x)dx, \qquad n \geq 0 \tag{23}$$

5. Queue Size Distribution at a Random Epoch

Defining the following probability generating functions

$$\left.\begin{aligned} P_q(x,z) &= \sum_{n=0}^{\infty} z^n P_n(x), & P_q(z) &= \sum_{n=0}^{\infty} z^n P_n, \\ V_q(x,z) &= \sum_{n=0}^{\infty} z^n V_n(x), & V_q(z) &= \sum_{n=0}^{\infty} z^n V_n, \\ D_q(x,z) &= \sum_{n=0}^{\infty} z^n D_n(x), & D_q(z) &= \sum_{n=0}^{\infty} z^n D_n \\ R_q(x,z) &= \sum_{n=0}^{\infty} z^n R_n(x), & R_q(z) &= \sum_{n=0}^{\infty} z^n R_n \\ C(z) &= \sum_{i=1}^{\infty} z^i c_i \end{aligned}\right\} \quad (24)$$

We Multiply equation (10) by z^n, take summation over n from 1 to ∞, adding to (11) then by simplifying and using equation (24) we get

$$\frac{\partial}{\partial x} P_q(x,z) + (\lambda - \lambda C(z) + \mu(x) + \alpha) P_q(x,z) = 0 \quad (25)$$

following similar process, from equations (12), (13), (14), (15), (17) and (18)we get respectively

$$\frac{\partial}{\partial x} V_q(x,z) + (\lambda - \lambda C(z) + \beta(x)) V_q(x,z) = 0 \quad (26)$$

$$\frac{\partial}{\partial x} D_q(x,z) + (\lambda - \lambda C(z) + \varphi(x)) D_q(x,z) = 0 \quad (27)$$

$$\frac{\partial}{\partial x} R_q(x,z) + (\lambda - \lambda C(z) + \gamma(x)) R_q(x,z) = 0 \quad (28)$$

Multiply equation (19) by z^{n+1}, sum over *n* from 0 to ∞, and use the generating functions defined in (24), we get

$$\begin{aligned} zP_q(0,z) = (1-p)\int_0^{\infty} P_q(x,z)\mu(x)dx + \int_0^{\infty} V_q(x,z)\beta(x)dx + \int_0^{\infty} R_q(x,z)\gamma(x)dx \\ + \lambda C(z)Q - \left[(1-p)\int_0^{\infty} P_0(x)\mu(x)dx + \int_0^{\infty} V_0(x)\beta(x)dx + \int_0^{\infty} R_0(x)\gamma(x)dx \right] \end{aligned} \quad (29)$$

using equation (16) to Replace

$-\left[(1-p)\int_0^{\infty} P_0(x)\mu(x)dx + \int_0^{\infty} V_0(x)\beta(x)dx + \int_0^{\infty} R_0(x)\gamma(x)dx \right]$ by $-\lambda Q$, we have

$$zP_q(0,z) = (1-p)\int_0^{\infty} P_q(x,z)\mu(x)dx + \int_0^{\infty} V_q(x,z)\beta(x)dx + \int_0^{\infty} R_q(x,z)\gamma(x)dx + \lambda Q(C(z)-1) \quad (30)$$

Now multiply equation (20) by z^n and sum over *n* from 1 to ∞, we get

$$V_q(0,z) = p\int_0^{\infty} P_q(x,z)\mu(x)dx \tag{31}$$

Similarly, we multiply equation (21) by z^n and sum over *n* from 0 to ∞, we get

$$D_q(0,z) = \alpha z P_q(z) \tag{32}$$

Now multiply equation (23) by z^n and sum over *n* from 0 to ∞, we get

$$R_q(0,z) = \int_0^{\infty} D_q(x,z)\varphi(x)dx \tag{33}$$

Integrating equation (25) from 0 to *x* yields

$$P_q(x,z) = P_q(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu(t)dt} \tag{34}$$

Where $P_q(0,z)$ is given by equation (30).

Integrating equation (34) by parts with respect to *x* yields

$$P_q(z) = P_q(0,z)\left[\frac{1-G^*(\lambda-\lambda C(z)+\alpha)}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{35}$$

Where $G^*(\lambda-\lambda C(z)+\alpha) = \int_0^{\infty} e^{-(\lambda-\lambda C(z)+\alpha)x} dG(x)$ is the Laplace-Stieltjes transform of the service time $G(x)$.

Now multiplying both sides of equation (34) by $\mu(x)$ and integrating over *x* we get

$$\int_0^{\infty} P_q(x,z)\mu(x)dx = P_q(0,z)G^*(\lambda-\lambda C(z)+\alpha) \tag{36}$$

Using equation (36) from equation (31) we get

$$V_q(0,z) = pP_q(0,z)G^*(\lambda-\lambda C(z)+\alpha) \tag{37}$$

Similarly, we integrate equation (26) from 0 to *x*, we get

$$V_q(x,z) = V_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \beta(t)dt} \tag{38}$$

Substituting by the value of $V_q(0,z)$ from (37) in equation (38) we get

$$V_q(x,z) = pP_q(0,z)G^*(\lambda-\lambda C(z)+\alpha)\ e^{-(\lambda-\lambda C(z))x-\int_0^x \beta(t)dt} \tag{39}$$

integrating equation (39) by parts with respect to *x* we get

$$V_q(z) = \frac{pP_q(0,z)G^*(\lambda-\lambda C(z)+\alpha)\left(1-B^*(\lambda-\lambda C(z))\right)}{(\lambda-\lambda C(z))} \tag{40}$$

Where $B^*[\lambda - \lambda C(z)] = \int_0^\infty e^{-(\lambda-\lambda C(z))x} dB(x)$ is the Laplace-Stieltjes transform of the vacation time $B(x)$.

Now multiplying both sides of equation (39) by $\beta(x)$ and integrating over x we get

$$\int_0^\infty V_q(x,z)\beta(x)dx = pP_q(0,z)G^*(\lambda - \lambda C(z) + \alpha)B^*(\lambda - \lambda C(z)) \tag{41}$$

Integrating equation (27) from 0 to x, yields

$$D_q(x,z) = D_q(0,z)e^{-(\lambda-\lambda C(z))x - \int_0^x \varphi(t)dt} \tag{42}$$

Substituting by the value of $D_q(0,z)$ from (32) in equation (42) we get

$$D_q(x,z) = \alpha z P_q(z) e^{-(\lambda-\lambda C(z))x - \int_0^x \varphi(t)dt} \tag{43}$$

integrating equation (43) by parts with respect to x we get

$$D_q(z) = \frac{\alpha z P_q(z)\left(1 - W^*(\lambda - \lambda C(z))\right)}{(\lambda - \lambda C(z))} \tag{44}$$

Where $W^*(\lambda - \lambda C(z)) = \int_0^\infty e^{-(\lambda-\lambda C(z))x} dW(x)$ is the Laplace-Stieltjes transform of the Delay time $W(x)$. Substituting the value $P_q(z)$ from equation (35) yields

$$D_q(z) = \frac{\alpha z P_q(0,z)\left(1 - G^*(\lambda - \lambda C(z) + \alpha)\right)\left(1 - W^*(\lambda - \lambda C(z))\right)}{(\lambda - \lambda C(z) + \alpha)(\lambda - \lambda C(z))} \tag{45}$$

Now multiplying both sides of equation (43) by $\varphi(x)$ and integrating over x we get

$$\int_0^\infty D_q(x,z)\varphi(x)dx = \alpha z P_q(z) W^*(\lambda - \lambda C(z)) \tag{46}$$

Substituting by the value of $P_q(z)$ from (35) in equation (46) we get

$$\int_0^\infty D_q(x,z)\varphi(x)dx = \frac{\alpha z P_q(0,z)\left(1 - G^*(\lambda - \lambda C(z) + \alpha)\right)W^*(\lambda - \lambda C(z))}{(\lambda - \lambda C(z) + \alpha)} \tag{47}$$

Now integrating equation (28) from 0 to x, yields

$$R_q(x,z) = R_q(0,z)e^{-(\lambda-\lambda C(z))x - \int_0^x \gamma(t)dt} \tag{48}$$

Substituting by the value of $R_q(0,z)$ from (33) and (47) in equation (48) we get

$$R_q(x,z) = \frac{\alpha z P_q(0,z)\left(1 - G^*(\lambda - \lambda C(z) + \alpha)\right)W^*(\lambda - \lambda C(z))}{(\lambda - \lambda C(z) + \alpha)} e^{-(\lambda-\lambda C(z))x - \int_0^x \gamma(t)dt} \tag{49}$$

integrating equation (49) by parts with respect to *x* we get

$$R_q(z)=\frac{\alpha z P_q(0,z)\left(1-G^*(\lambda-\lambda C(z)+\alpha)\right)W^*(\lambda-\lambda C(z))(1-H^*(\lambda-\lambda C(z))}{(\lambda-\lambda C(z)+\alpha)(\lambda-\lambda C(z))} \quad (50)$$

Now multiplying both sides of equation (49) by $\gamma(x)$ and integrating over *x* we get

$$\int_0^\infty R_q(x,z)\gamma(x)dx=\frac{\alpha z P_q(0,z)\left(1-G^*(\lambda-\lambda C(z)+\alpha)\right)W^*(\lambda-\lambda C(z))H^*(\lambda-\lambda C(z))}{(\lambda-\lambda C(z)+\alpha)} \quad (51)$$

Assuming $a=(\lambda-\lambda C(z)+\alpha),\ b=(\lambda-\lambda C(z))$

Now using equations (36), (41) and (51) in equation (30) yields

$$P_q(0,z)=\frac{-abQ}{a\left(z-G^*(a)\left(1-p+pB^*(b)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)} \quad (52)$$

Substituting $P_q(0,z)$ in equation (35), (3.40), (45) and (50) we get

$$P_q(z)=\frac{-bQ(1-G^*(a))}{a\left(z-G^*(a)\left(1-p+pB^*(b)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)} \quad (53)$$

$$V_q(z)=\frac{-apQG^*(a)\left(1-B^*(b)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(b)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)} \quad (54)$$

$$D_q(z)=\frac{-Q\alpha z\left(1-G^*(a)\right)\left(1-W^*(b)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(b)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)} \quad (55)$$

$$R_q(z)=\frac{-Q\alpha z\left(1-G^*(a)\right)W^*(b)(1-H^*(b)}{a\left(z-G^*(a)\left(1-p+pB^*(b)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)} \quad (56)$$

Let $S_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. i.e $S_q(z)=P_q(z)+V_q(z)+D_q(z)+R_q(z)$

Then adding equations (53), (54), (55) and (56) we obtain

$$S(z)=\frac{-Q(1-G^*(a))\left\{b+\alpha z(1-W^*(b)H^*(b))\right\}-apQG^*(a)\left(1-B^*(b)\right)}{a\left(z-G^*(a)\left(1-p+pB^*(b)\right)\right)-\alpha z\left(1-G^*(a)\right)W^*(b)H^*(b)} \quad (57)$$

In order to find *Q*, we use the normalization condition

$$S_q(1) + Q = 1$$

Note that for $z = 1$, $S_q(1)$ is indeterminate of 0/0 form. Therefore, we apply L'Hopitals Rule on equation (57), we get

$$S(1) = \frac{\lambda E(I)Q(1-G^*(\alpha))\left(1+\alpha E(R)+\alpha E(D)\right)+ p\alpha\lambda E(I)QE(V)G^*(a)}{\alpha\left(1- p(E(V)(\lambda E(I)G^*(\alpha)\right)-\lambda E(I)\left(1-G^*(\alpha)\right)\left(1+\alpha E(R)+\alpha E(D)\right)-\alpha\left(1-G^*(\alpha)\right)} \quad (58)$$

Therefore, adding Q to equation (58)and equating to 1 and simplifying we get

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)-E(D)+p(E(V)\right) \quad (59)$$

Equation (59) gives the probability that the server is idle. From equation (59) the utilization factor, ρ of the system is given by

$$\rho = \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{E(R)}{G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}-\frac{1}{\alpha}-E(R)-E(D)+p(E(V)\right) \quad (60)$$

where $\rho < 1$ is the stability condition under which the steady states exits. Substituting for Q from (59) into (57), we have completely and explicitly determined $S_q(z)$, the probability generating function of the queue size.

6. Particular Cases

Case1. Exponential repair time.

The most common distribution function for the repair time the exponential distribution. For this distribution, the rate of service $\gamma > 0$, and we have

$$H^*(b) = \frac{\gamma}{b+\gamma}, \qquad E(R) = \frac{1}{\gamma}, \quad (61)$$

using (61) in equations (57), (59) and (60) respectively we get

$$S_q(z) = \frac{-Q(1-G^*(a))\left(cb+\alpha z(c-\gamma W^*(b))\right)-acpQG^*(a)\left(1-B^*(b)\right)}{a\,c\left(z-G^*(a)\left(1-p+pB^*(b)\right)\right)-\alpha\gamma z\left(1-G^*(a)\right)W^*(b)} \quad (62)$$

$$Q = 1-\lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)}+\frac{1}{\gamma G^*(\alpha)}+\frac{E(D)}{G^*(\alpha)}-\frac{1}{\alpha}-\frac{1}{\gamma}-E(D)+p(E(V)\right) \quad (63)$$

$$\rho = \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{1}{\gamma G^*(\alpha)} + \frac{E(D)}{G^*(\alpha)} - \frac{1}{\alpha} - \frac{1}{\gamma} - E(D) + p(E(V)) \right) \tag{64}$$

The results obtained in (62), (3.63) and (64) agree with the result given by the same authors in previous study (to be published).

Case 2. No Delay for Repairs to Start

Once the system breakdown, if its repairs start immediately and there is no delay time we let $E(D) = 0$ and $W^*(b) = 1$ Using this in the main results of this paper, from equations (57),

$$S(z) = \frac{-Q(1-G^*(a))\{b + \alpha z(1-H^*(b))\} - apQG^*(a)\left(1-B^*(b)\right)}{a\left(z - G^*(a)\left(1 - p + pB^*(b)\right)\right) - \alpha z\left(1-G^*(a)\right)H^*(b)} \tag{65}$$

$$Q = 1 - \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{E(R)}{G^*(\alpha)} - \frac{1}{\alpha} - E(R) + p(E(V)) \right) \tag{66}$$

$$\rho = \lambda E(I)\left(\frac{1}{\alpha G^*(\alpha)} + \frac{E(R)}{G^*(\alpha)} - \frac{1}{\alpha} - E(R) + p(E(V)) \right) \tag{67}$$

The results obtained in (65), (66) and (67) agree with the result given by Maraghi, Madan, & Darby-Dowman,(2009a).

7. The Average Queue Size and the Average Waiting Time

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} S_q(z)\Big|_{z=1} \tag{68}$$

Since this formula gives 0/0 form, then we write $S_q(z)$ given in (57) as $S_q(z) = N(z)/D(z)$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (3.57) respectively. Then using the L'Hopital's rule twice we obtain

$$L_q = \lim_{z\to 1} \frac{D'(z)N''(z) - N'(z)D''(z)}{2\big(D'(z)\big)^2} \tag{69}$$

where primes and double primes in (69) denote first and second derivatives at z = 1, respectively. Carrying out the derivatives at z = 1 we have

$$N'(1) = Q\lambda E(I)\Big((1-G^*(\alpha))\big(1+\alpha(E(D)+E(R))\big)+\alpha pE(V)G^*(\alpha)\Big) \tag{70}$$

$$\begin{aligned} N''(1) &= Q\alpha(\lambda E(I))^2(1-G^*(a))\big[E(D^2)+E(R^2)+E(D)+E(R)+2E(D)E(R)\big] \\ &+ Q(1-G^*(a))\big[\lambda E(I)\big(E(D)-E(R)\big)\big(1+\alpha\big)\big]+Q\lambda E(I(I-1))\Big(1-G^*(a)\big(1-p\alpha E(V)\big)\Big) \\ &+ 2QG^{*'}(\alpha)(\lambda E(I))^2\big[1-\alpha\big(E(D)+E(R)\big)\big] \\ &- 2pQ(\lambda E(I))^2E(V)\Big[G^*(\alpha)+\alpha G^{*'}(\alpha)\Big]+pQ\alpha G^*(\alpha)E(V^2)(\lambda E(I))^2 \end{aligned} \tag{71}$$

$$D'(1) = -\alpha p(E(V)(\lambda E(I)G^*(\alpha)-\lambda E(I)\big(1-G^*(\alpha)\big)\big(1+\alpha E(R)+\alpha E(D)\big)+\alpha G^*(\alpha) \tag{72}$$

$$\begin{aligned} D''(1) &= \alpha p(\lambda E(I))^2\Big(2G^{*'}(\alpha)E(V)-G^*(\alpha)E(V^2)\Big)-\alpha G^*(\alpha)pE(V)\lambda E(I(I-1)) \\ &- 2\lambda E(I)\Big(1-G^*(a)pE(V)\lambda E(I)+G^{*'}(a)(\lambda E(I)+\alpha)\Big) \\ &- \alpha\big(1-G^*(a)\big)\big\{\big[E(R)+E(D)\big]\big(\lambda E(I(I-1))+\lambda E(I)\big) \\ &+ (\lambda E(I))^2\big[2E(R)E(D)+E(R^2)+E(D^2)\big]\big\} \\ &- \alpha\lambda E(I)\Big(G^{*'}(\alpha)\lambda E(I)+1-G^*(\alpha)\Big)\big(E(D)+E(R)\big) \\ &- \lambda E(I(I-1))\big(1-G^*(a)\big) \end{aligned} \tag{73}$$

Where $E(V^2)$ is the second moment of the vacation time, $E(R^2)$ is the second moment of the repair time $E(D^2)$ is the second moment of the Delay time, $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers, and Q has been found in (59).

Then if we substitute the values of $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ from (70), (71), (72), and (73) into (69) we obtain L_q in a closed form. Further, the mean waiting time of a customer could be found using Little's Law formula $W_q = L_q/\lambda$.

8. Numerical Example

To illustrate the results of this chapter numerically we consider the service times, vacation times, delay times and repair times are exponentially distributed. All the values were chosen so that the steady state condition is satisfied. In **Table 1** We choose the following values:
$\mu = 7,\ \beta = 5,\ \gamma = 3,\ \lambda = 2,\ \varphi = 8,\ E(I) = 1$ and $E(I(I-1)) = 0$, we consider that α takes the values 1, 2 and 3,while p takes the values 0.25, 0.5 and 0.75.

Table 1

p	α	ρ	Q	Lq	Wq	L	W
	1	0.5167	0.4833	1.3795	0.6898	1.8962	0.9481
0.25	2	0.6476	0.3524	2.4333	1.2167	3.081	1.5405
	3	0.7786	0.2214	4.8331	2.4166	5.6117	2.8059
	1	0.6167	0.3833	1.8441	0.9221	2.4608	1.2304
0.5	2	0.7476	0.2524	3.5835	1.7917	4.3311	2.1656
	3	0.8786	0.1214	9.2496	4.6248	10.1282	5.0641
	1	0.7167	0.2833	2.6367	1.3183	3.3533	1.6767
0.75	2	0.8476	0.1524	6.2432	3.1216	7.0908	3.5454
	3	0.9786	0.0214	54.8869	27.4435	55.8655	27.9327

Table 1 shows that increasing the value of α or p causes increasing the value of utilization factor, the mean queue size and the mean waiting time of the customers while the server idle time decreases.
In **Table 2** we consider we consider
$\mu = 7,\ \lambda = 2, \alpha = 2,\ \varphi = 8, p = 0.25\, E(I) = 1$ and $E(I(I-1)) = 0$, while β takes the values 2, 5, 8 and γ takes the values 3, 5, and 7.

Table 2

γ	β	ρ	Q	Lq	Wq	L	W
	2	0.7976	0.2024	5.326	2.663	6.1236	3.0618
3	5	0.6476	0.3524	2.4333	1.2167	3.081	1.5405
	8	0.6101	0.3899	2.1301	1.065	2.7402	1.3701
	2	0.7214	0.2786	3.0786	1.5393	3.8	1.9
5	5	0.5714	0.4286	1.4757	0.7378	2.0471	1.0235
	8	0.5339	0.4661	1.2965	0.6482	1.8304	0.9152
	2	0.6888	0.3112	2.4875	1.2437	3.1762	1.5881
7	5	0.5388	0.4612	1.1858	0.5929	1.7246	0.8623
	8	0.5013	0.4987	1.0392	0.5196	1.5404	0.7702

Table 2 shows that increasing the value of β or γ causes decreasing the value of utilization factor, the mean queue size and the mean waiting time of the customers while the server idle time increases.
In **Table 3** we consider
$\mu = 7,\ \ \lambda = 2, \alpha = 2,\ \beta = 5, p = 0.25,\ \gamma = 3, E(I) = 1$ and $E(I(I-1)) = 0$, while φ varies from 3 to 11.

Table 3

φ	ρ	Q	Lq	Wq	L	W
3	0.7667	0.2333	5.4549	2.7274	6.2215	3.1108
4	0.719	0.281	3.9198	1.9599	4.6389	2.3194
5	0.6905	0.3095	3.2366	1.6183	3.9271	1.9636
6	0.6714	0.3286	2.8516	1.4258	3.523	1.7615
7	0.6578	0.3422	2.6048	1.3024	3.2626	1.6313
8	0.6476	0.3524	2.4333	1.2167	3.081	1.5405
9	0.6397	0.3603	2.3073	1.1537	2.947	1.4735
10	0.6333	0.3667	2.2108	1.1054	2.8442	1.4221
11	0.6281	0.3719	2.1346	1.0673	2.7627	1.3814

It is clear from Table 3 that as long as we increase the value of φ, the server idle time increases while the utilization factor, the mean queue size and the mean waiting time of the customers, all decrease. All the trends shown by the tables are as expected.

References

[1] Alfa, A. S. 'Vacation models in discrete time', Queueing Systems, 44 (1), (2003), 5-30.

[2] Avi-Itzhak, B. & Naor, P. 'Some queueing problems with the service station subject to breakdown', Operations Research, 11 (3), (1963), 303-320.

[3] Bunday, B. D. An introduction to queueing theory. Oxford, England: Oxford University Press,1996.

[4] Choi, B. D., Hwang, G. U. & Han, H. D. 'Supplementary variable method applied to the MAP/G/1 queueing systems', Journal of Australian Mathematical Society – Series B, 40, (1998), 86–96.

[5] Choudhury, G. 'An M/G/1 queue with an optional second vacation', International Journal of Information and Management Sciences, 17 (3), (2006), 19- 30.

[6] Cox, D. 'The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables', Proceedings of the Cambridge Phylosophical Society, 51, (1955), 433-441.

[7] Hur, S. & Ahn, S. 'Batch arrival queues with vacations and server setup', Applied Mathematical Modelling, 29, (2005), 1164-1181.

[8] Kashyap, B. R. & Chaudhry, M. L. An Introduction to Queueing Theory. Ontario, Canada: A & A Publications, 1988.

[9] Kulkarni, V. G. & Choi, B. D. 'Retrial queues with server subject to breakdowns and repairs', Queueing Systems, 7 (2), (1990), 191-209.

[10] Madan, K. C. & Abu Al-Rub, A. Z. 'On a single server queue with optional phase type server vacations based on exhaustive deterministic service and a single vacation policy', Applied Mathematics and Computation, 149, (2004), 723-734.

[11] Madan, K. C. & Al-Rawwash, M. 'On the Mx/G/1 queue with feedback and optional server vacations based on a single vacation policy', Applied Mathematics and Computation, 160, (2005), 909-919.

[12] Madan, K. C. & Choudhury, G. 'Steady state analysis of an MX/(G1,G2)/1 queue with restricted admissibility and random setup time', International Journal of Information and Management Sciences, 17 (2), (2006), 33-56.

[13] Madan, K. C. 'On a M[x]/M[b]/1 queueing system with general vacation times', International Journal of Information and Management Sciences, 2 (1), (1991), 51-60.

[14] Madan, K. C, 'An M/G/1 queue with second optional server', Queueing Systems, 34 (1-4), . (2000a). 37-46.

[15] Madan, K. C., Abu-Dayyeh, W. & Gharaibeh, M. 'Steady state analysis of two Mx/Ma,b/1 queue models with random breakdowns', International Journal of Information and Management Sciences, 14 (3), (2003b), 37-51.

[16] Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. 'Batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general vacation time distribution', International Journal of Information and Management Sciences, 20 (1), (2009a), 55-70.

[17] Takine, T. & Sengupta, B. 'A single server queue with service interruptions', Queueing Systems, 26 (3-4), (1997), 285-300.

[18] Wang, J., Cao, J. & Li, Q. 'Reliability analysis of the retrial queue with server breakdowns and repairs', Queueing Systems, 38 (4), (2001), 363-380.

Received: July, 2010