

Efficiency of Inhomogeneous Thermoelectric Generators

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Abstract

We revisit the classical problem of the efficiency of inhomogeneous thermoelectric generators. The effects of different physical parameters on the efficiency of a generator are studied by solving the Domenicali's equation numerically. It is found that the efficiency of a thermoelectric generator is insensitive to both the electrical resistivity and thermal conductivity. However, the efficiency of a thermoelectric generator can be improved by various designs of the Seebeck coefficient.

Mathematics Subject Classification: 34B60, 80A20

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1 Introduction

The thermoelectric effect refers to the direct conversion of temperature differences to electrical power and vice versa [1, 3]. This effect can be used to generate electricity, to measure temperature, to cool or heat objects. For example, some spaceships are powered in this way, exploiting the temperature difference between a radioactively-heated plate and the cold empty space surrounding the ship. Future work in thermoelectrics includes converting waste heat from power plants, trucks and even automobiles into electricity.

There are many advantages to thermoelectrics. For example, thermoelectric material devices are extremely reliable and are capable of over 11 years of steady state operation because the devices contain no moving parts. Unfortunately, a major obstacle to wide application of thermoelectric energy conversion is its low efficiency.

In [2] Mahan carried out theoretical calculations which showed that inhomogeneous doping can increase the efficiency of thermoelectric generators.

His study was based on a special set of equations which defined the transport coefficients in terms of a normalized conductivity. In this paper we revisit the efficiency of thermoelectric generators by isolating each parameter and studying its effect on the efficiency directly. Our approach is different from [2] in the way that each parameter is studied individually and therefore one can gain more insights on how to improve the efficiency of thermoelectric generators.

This paper is organized in the following way. First, we briefly review the governing equations for a one-dimensional thermoelectric transport. Then, we solve the system numerically and present our numerical results. Finally, we draw conclusions.

2 Governing equations for thermoelectric generators

For a one-dimensional thermoelectric transport where the current is assumed to flow in one dimension, the governing equations consist of the Domenicali's equation for the steady state energy balance

$$\frac{d}{dx}\left[K(x)\frac{dT(x)}{dx}\right] = -\rho(x)J^2 + JT(x)\frac{dS(x)}{dx}, \quad (1)$$

and the definition of the heat flow

$$Q(x) = JT(x)S(x) - K(x)\frac{dT(x)}{dx}. \quad (2)$$

Here K denotes the thermal conductivity, T the temperature, ρ the electrical resistivity, J the electrical current, S the Seebeck coefficient, and Q the heat flow.

Note that equation (1) is a second order ODE. In order to get a first order ODE system, we manipulate the equations (1) and (2) as follows. Differentiating (2) with respect to x yields

$$\frac{dQ(x)}{dx} = JT'(x)S(x) + JT(x)S'(x) - \left[K(x)\frac{dT(x)}{dx}\right]'. \quad (3)$$

Substituting the expression $\frac{d}{dx}\left[K(x)\frac{dT(x)}{dx}\right]$ from (1) into (3), we obtain

$$\frac{dQ(x)}{dx} = J(x)T'(x)S(x) + \rho(x)J^2. \quad (4)$$

Solving (2) for $T'(x)$ gives

$$T'(x) = \frac{JT(x)S(x) - Q(x)}{K(x)}. \quad (5)$$

Plugging (5) into (4), we have

$$\frac{dQ(x)}{dx} = \frac{J^2 T(x) S^2(x)}{K(x)} - \frac{JS(x)Q(x)}{K(x)} + \rho(x)J^2. \quad (6)$$

Introducing $Z = \frac{S^2}{\rho K}$, equation (6) becomes

$$\frac{dQ(x)}{dx} = \rho(x)J^2[1 + Z(x)T(x)] - \frac{JS(x)Q(x)}{K(x)}. \quad (7)$$

Putting equations (5) and (7) together, we finally arrive at a first order ODE system:

$$\begin{cases} \frac{dT(x)}{dx} = \frac{JT(x)S(x) - Q(x)}{K(x)} \\ \frac{dQ(x)}{dx} = \rho(x)J^2[1 + Z(x)T(x)] - \frac{JS(x)Q(x)}{K(x)} \end{cases} \quad (8)$$

The boundary conditions are imposed such that there is a cold temperature at one end of the bar (which is set to be origin for convenience) and a hot temperature at the other end of the bar. Mathematically, this implies that

$$T(0) = T_c, \quad T(L) = T_h \quad \text{where } T_c < T_h. \quad (9)$$

In the special case where K , S and ρ are constants, the above BVP can be solved exactly. First, (1) is reduced to

$$\frac{d^2 T(x)}{dx^2} = -\frac{\rho J^2}{K}. \quad (10)$$

Upon integration twice, we find $T(x) = -\frac{\rho J^2}{2K}x^2 + C_1x + C_2$. The boundary conditions (9) require that $C_1 = \frac{T_h - T_c}{L} + \frac{\rho J^2 L}{2K}$, $C_2 = T_c$. Thus, the solution of Domenicali's equation (1) is

$$T(x) = \frac{\rho J^2}{2K}x(L-x) + \frac{T_h - T_c}{L}x + T_c. \quad (11)$$

Consequently, the corresponding solution for $Q(x)$ can be obtained directly from (2):

$$Q(x) = JST(x) - \frac{K(T_h - T_c)}{L} - \frac{\rho J^2}{2}(L - 2x). \quad (12)$$

In general, the parameters vary along the bar and one has to solve the system (8) with two boundary conditions (9) numerically. A MATLAB build-in function BVP4C can be used directly to solve this BVP. BVP4C is a finite difference code that implements the three-stage Lobatto IIIA formula and solves the boundary value problems for ODEs by collocation. The basic idea of a

collocation method is to first select a family of candidate solutions (usually polynomials up to a certain degree) and a number of points (called collocation points) in the computational domain $[a, b]$, and then to find the solution so that the given equation is satisfied by this solution at the collocation points. The collocation polynomial of BVP4C provides a solution that is continuous on $[a, b]$ and has a continuous first derivative there. The solution is fourth-order accurate uniformly in $[a, b]$. More information on BVP4C can be found in the book [4].

The efficiency of a thermoelectric generator is defined as the ratio of the output power and the input heat flow:

$$\eta = \frac{J \int_0^L S(x) \frac{dT(x)}{dx} dx + J^2 \int_0^L \rho(x) dx}{Q_h}, \quad (13)$$

where Q_h is the heat flow at the hot end of the bar: $Q_h = Q(L)$. In the simplest case where K , S and ρ are constants, (13) is reduced to

$$\eta = \frac{JS(T_h - T_c) + J^2 \rho L}{JST_h - \frac{K(T_h - T_c)}{L} + \frac{J^2 \rho L}{2}}, \quad (14)$$

where equation (12) has been exploited to evaluate Q_h .

To compute (13) numerically, we first apply the cubic spline to interpolate the solution values obtained from BVP4C and then use the Composite Simpson's Rule to evaluate the integration.

3 Numerical results

For the numerical solutions of BVP (8) and (9), we consider several cases below. In order to pinpoint the effect of each parameter on the efficiency, we will vary one parameter at a time and compare the solutions with the constant parameter case.

3.1 Effect of the Seebeck coefficient

In order to investigate the effect of the Seebeck coefficient on the efficiency of a thermoelectric generator, we study various profiles of $S(x)$.

Case 1: For comparison purpose, we consider first the simplest case where K , S and ρ are constants. Similar to [2], we choose $T_c = 400$, $T_h = 750$, $\rho = 0.04$, $K = 3.194$, $S = 211$, $J = -1.23$, $L = 1$. Figure 1 shows both the exact solutions (11) and (12) and the numerical solutions. The two solutions agree with each other very well and this validates the numerical solutions. The efficiency in this case can be found either from (14) or numerically to be 0.4640.

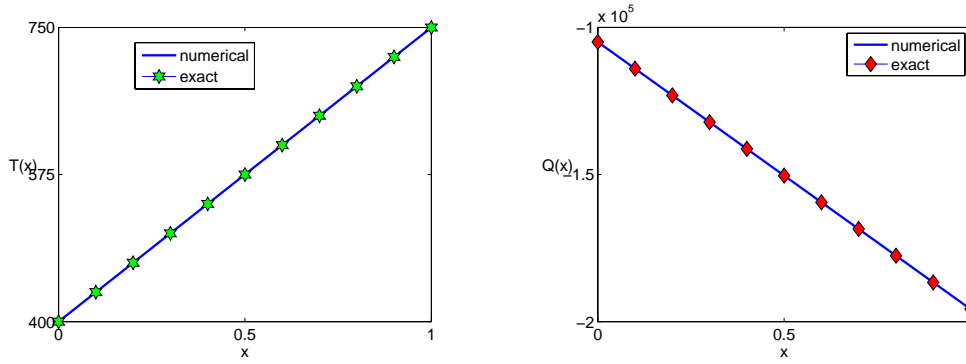


Figure 1: Comparison of the numerical solution and the exact solution. The efficiency is 0.4640.

Case 2: Now we allow the parameters to vary spatially along the bar, which corresponds to inhomogeneous materials. Instead of assuming the parameters as functions of temperature T [2], we treat the parameters directly as functions of x . First, we assume $S(x) = s_0(1 + x)$ where $s_0 = 211$ and keep all other parameters same as in Case 1. In Figure 2 we compare the solutions of this inhomogeneous case and the homogeneous case in Case 1. Unlike the homogeneous case where the temperature and the heat flow change almost linearly with x , the inhomogeneous solutions behave more dramatically as functions of x and the efficiency is decreased to 0.4055.

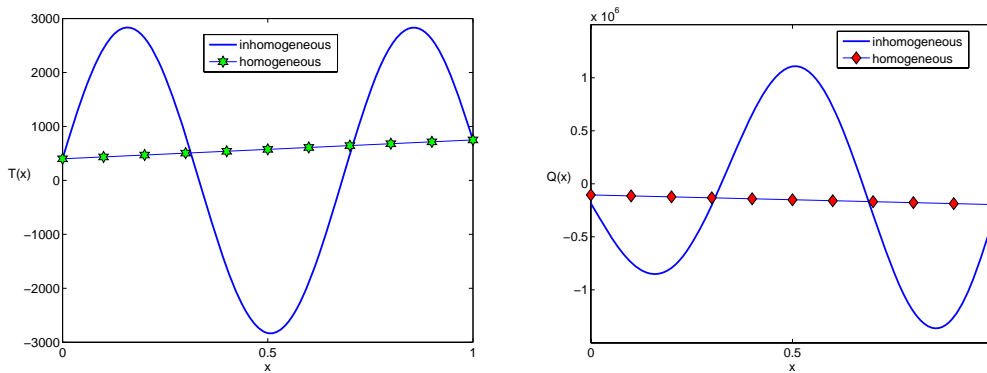


Figure 2: Comparison of the inhomogeneous solution and the homogeneous solution. The inhomogeneous case corresponds to $S(x) = s_0(1 + x)$. The efficiency for the inhomogeneous case is 0.4055 whereas the efficiency for the homogeneous case is 0.4640.

Case 3: In Case 3 we vary S as $S(x) = s_0(1 + x^2)$ with $s_0 = 211$ and keep all the other parameters unchanged from Case 1. Figure 3 compares the

inhomogeneous solution with the homogeneous solution. Again, large oscillations of the temperature and the heat flow are observed and the efficiency is increased significantly to 0.6064.

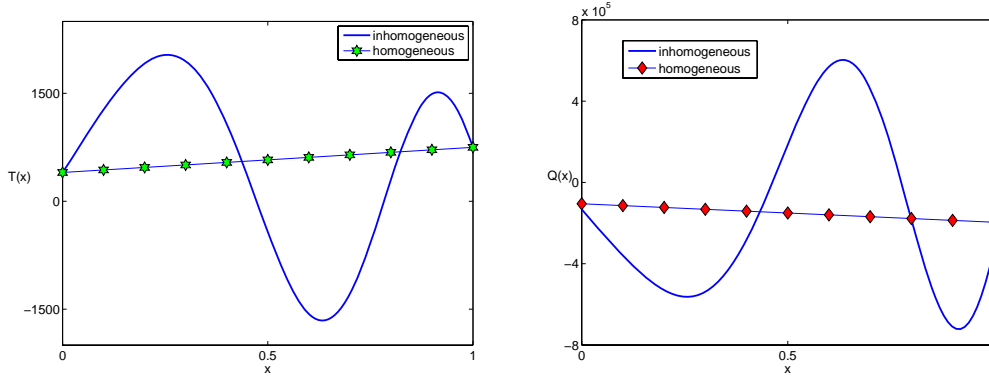


Figure 3: Comparison of the inhomogeneous solution and the homogeneous solution. The inhomogeneous case corresponds to $S(x) = s_0(1 + x^2)$. The efficiency for the inhomogeneous case is 0.6064 whereas the efficiency for the homogeneous case is 0.4640.

Case 4: Assume $S(x) = s_0(1 + \sin x)$ where $s_0 = 211$ and all other parameters retain the same values as in Case 1. The solutions are depicted in Figure 4. The efficiency is increased dramatically to the value 0.6218. This is about 34% increase from the homogeneous case.

Case 5: We consider $S(x) = s_0(1 + \cos x)$ where $s_0 = 211$ and all the other parameters have the same values as in Case 1. Figure 5 plots the solutions. The efficiency is decreased to the value 0.3617.

Case 6: Let $S(x) = s_0(1 + e^x)$ where $s_0 = 211$. Again, all other parameters are kept at the same values as in Case 1. The solutions are shown in Figure 6. The efficiency is jumped to 0.8516, which is about 83% increase from the homogeneous case.

From these case studies, it is clear that the efficiency of a thermoelectric generator can be improved significantly by inhomogeneous doping with a smart choice of the Seebeck coefficient. Whether or not it is feasible in practice remains a challenge.

3.2 Effect of thermal conductivity

Now we keep the Seebeck coefficient S as a constant and vary the thermal conductivity K . All the constant parameters are taken the same values as in Case 1. Both Figure 7 and Figure 8 show the solutions corresponding to various profiles of $K(x)$. Figure 7 indicates that the larger values of $K(x)$ tend

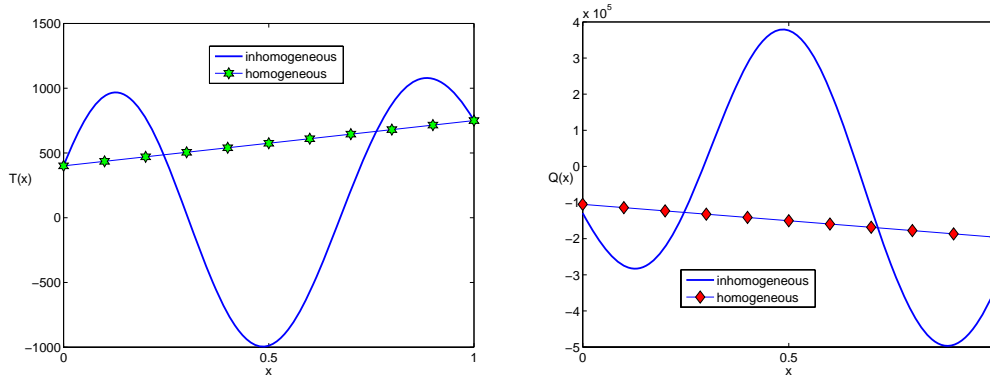


Figure 4: Comparison of the inhomogeneous solution and the homogeneous solution. The inhomogeneous case corresponds to $S(x) = s_0(1 + \sin x)$. The efficiency for the inhomogeneous case is 0.6218 whereas the efficiency for the homogeneous case is 0.4640.

to reduce the efficiency while Figure 8 suggests that the smaller values of $K(x)$ increase the efficiency. However, from these two figures we can see that the efficiency is rather insensitive to different profiles of $K(x)$.

3.3 Effect of electrical resistivity

Finally, we fix all the other parameters and let $\rho(x)$ vary. We find that both the solution and the efficiency are very insensitive to variations in $\rho(x)$ even though it undergoes a large change of magnitudes. We omit the plots of solutions here.

4 Conclusions

The efficiency of a thermoelectric generator is studied here by varying different physical parameters separately. It is found that the efficiency of a thermoelectric generator can be increased significantly by selecting an inhomogeneous Seebeck coefficient. On the other hand, the efficiency is rather insensitive to the variations of the electrical resistivity and thermal conductivity. These results serve as a first step towards improving the efficiency of a thermoelectric generator. The design of an optimal Seebeck coefficient for the thermoelectric generator will be pursued in the future.

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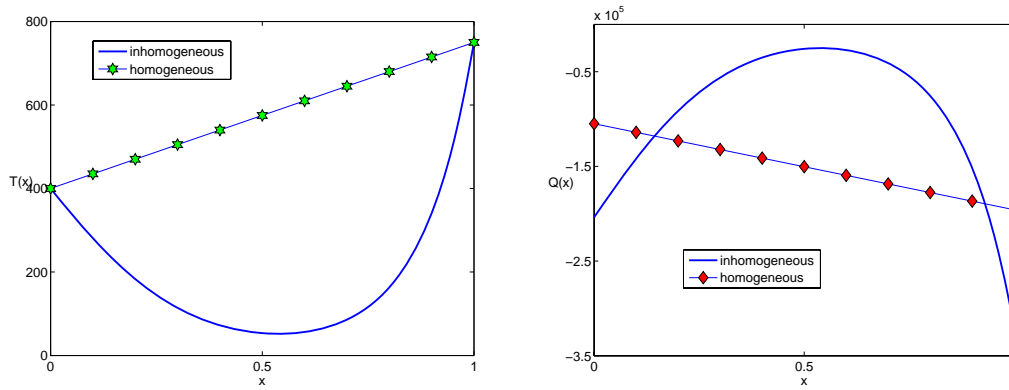


Figure 5: Comparison of the inhomogeneous solution and the homogeneous solution. The inhomogeneous case is given by $S(x) = s_0(1 + \cos x)$. The efficiency for the inhomogeneous case is 0.3617 whereas the efficiency for the homogeneous case is 0.4640.

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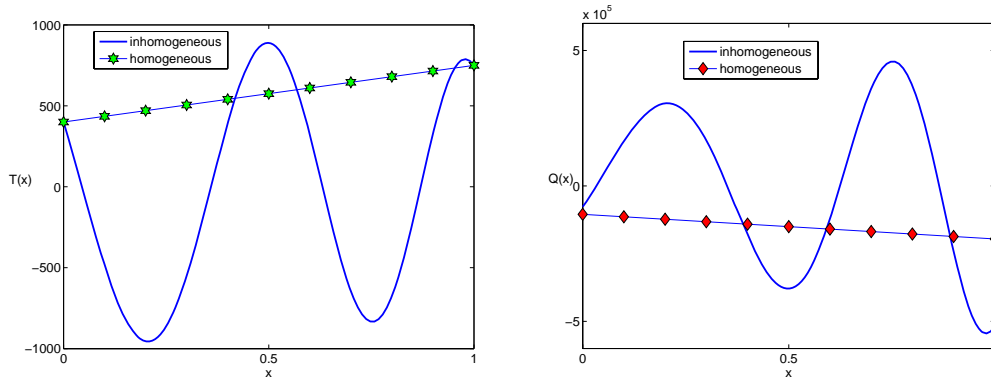


Figure 6: Comparison of the inhomogeneous solutions and the homogeneous solutions. The inhomogeneous case corresponds to $S(x) = s_0(1 + e^x)$. The efficiency for the inhomogeneous case is 0.8516 whereas the efficiency for the homogeneous case is 0.4640.

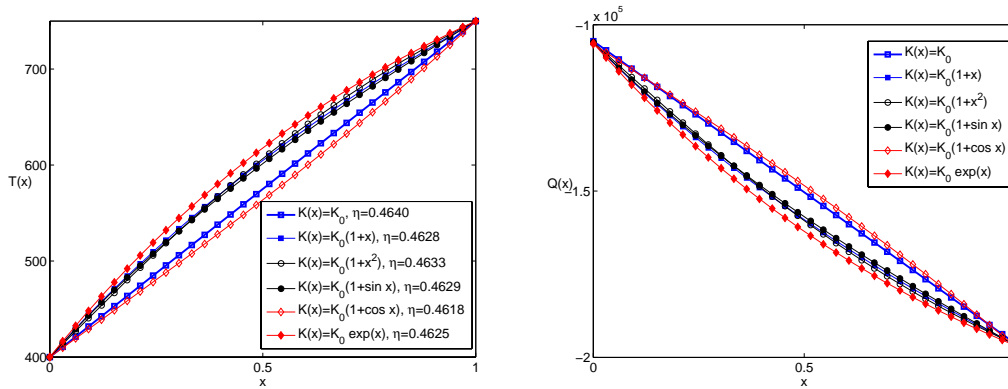


Figure 7: Solutions of various inhomogeneous cases where the thermal conductivity $K(x)$ varies.

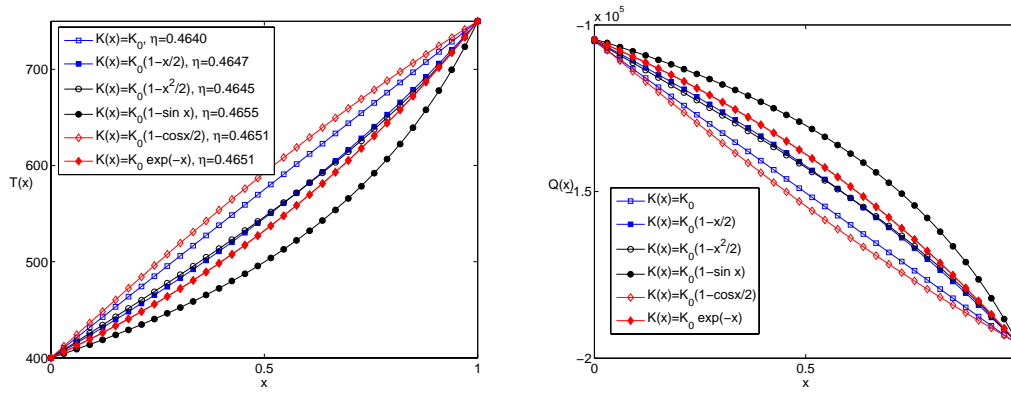


Figure 8: Solutions of various inhomogeneous cases by changing the profile of the thermal conductivity.