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Sumudu Transform Applications to Bessel Functions and Equations

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Abstract

This paper starts by presenting a group of transforms that act on power series, having *the Sumudu* as a generating prototype. Among others, Sumudu convolution and shift properties are examined, extended, and used to treat Bessel equations. This relatively new transform usage is then defended in light of the bilateral Laplace-Sumudu duality.

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0. Motivation: An Interesting Group of Transforms.

For any real number p, consider the set of parametrized power series,

$$f_p(t) = \sum_{k=0}^{\infty} t^k/[k!]^p, \tag{0.1}$$

then, $f_1(t) = e^t$, $f_0(t)$, is the geometric series converging to the function, $1/(1-t)$, in the time interval $(-1, 1)$, and all f_p are *formally* smoothly defined in $\mathbb{R}$. For $p > 0$, the tail end of the power series, $f_p(t)$, decreases rapidly the larger the value p is, and $f_p(t)$ tends to the null function as p tends to infinity. For $p < 0$, the power series tail end grows very fast, and f_p fails to converge except at $t = 0$.

If $f_p(t)$ functionally describes an electromagnetic or power signal, then the value p, has a direct bearing on the the strength of that signal. The signal fades away infintesimally the larger the value p gets, and will intensify to unbounded proportions, as p decreases to $-\infty$. Reaching such extremes, tends to render physical observations, and mathematical manipulations practically inadequate, if not impossible. This highly motivates the inception of tools, that map the signal to a more captable and manageable version. In this section, we introduce a group of operators that do just that.

Consider the operator, $\mathbb{S}$, mapping the series, $\sum_{k=0}^{\infty} a_k x^k$, into the series, $\sum_{k=0}^{\infty} b_k v^k$,

$$\mathbb{S}[f(x)](v) = \mathbb{S}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \mathbb{S}[a_k x^k](v) = \sum_{k=0}^{\infty} k! a_k v^k = \sum_{k=0}^{\infty} b_k v^k = g(v). \tag{0.2}$$

Clearly, if $f^{\,(k)}$, denotes differentiating the power series k times, then, $g(0) = f(0)$, and, $g^{(k)}(0) = k! f^{\,(k)}(0)$. The tranform, $\mathbb{S}$, is linear, fixes linear functions, and preserves units. This means that x and v may be interchanged, albeit the domain of definition of the power series, f and g may differ. In addition, $\mathbb{S}[f_1\,(x)] = f_0(v)$, and for any p, we have,

$$\mathbb{S}[f_p(x)] = f_{(p\,-1)}(v). \tag{0.3}$$

In fact, given any real power q, we may then define the operator, $\mathbb{S}^q$, acting on power series as follows,

$$\mathbb{S}^q \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \mathbb{S}^q[a_k x^k] = \sum_{k=0}^{\infty} a_k (k!)^q v^k. \tag{0.4}$$

Hence, for any reals p, q,and λ, we can apply powers of the transform, $\mathbb{S}$, to the parametrized family of functions in equation (0.1) to get,

$$\mathbb{S}^q[f_p(\lambda x)] = f_{(p\,-q)}(\lambda v). \tag{0.5}$$

The operator, $\mathbb{S}^q$, does not change the nature of the initial units, albeit it may have a warping effect on the domain of the image. The convergence of the resulting power series to an explicit function, may not be possible, since the image region of convergence may be empty or the singleton, $\{0\}$. In such as case, equation (0.4) remains true only in the formal sense (symbolwise), since there is no explicit function to which the right hand power $v-$series converges. The set, $\{\mathbb{S}^q$, q is real$\}$, operating on formal powers series such as in (0.1), along with compostion, form a commutative group with the operator, $\mathbb{S}^0 = \mathbb{I}$, for an identity, where the inverse of any given operator, $\mathbb{S}^q$, is given by, $(\mathbb{S}^q)^{-1} = \mathbb{S}^{-q}$, and such that for all real pairs, (q_1,q_2),

$$\mathbb{S}^{q_1}\mathbb{S}^{q_2} = \mathbb{S}^{q_2}\mathbb{S}^{q_1} = \mathbb{S}^{q_1+q_2} = \mathbb{S}^{q_2}\mathbb{S}^{q_1}, \tag{0.6}$$

and,

$$(\mathbb{S}^{q_1})^{q_2} = \mathbb{S}^{q_2 q_1} = \mathbb{S}^{q_2 q_1} = (\mathbb{S}^{q_2})^{q_1}. \tag{0.7}$$

Furthermore, the operators, $\mathbb{S}^q$, are obviously linear,

$$\mathbb{S}^q \left[af(t) + bg\left(t\right)\right] = a\mathbb{S}^q \left[f\left(t\right)\right] + b\mathbb{S}^q \left[g\left(t\right)\right]. \tag{0.8}$$

It can be easily verified that the function $\mathbb{S}^q \left[f(x)\right]$ keeps the same units as $f(x)$. In fact, we can easily check that for any real (or complex) number λ,

$$\mathbb{S}^q[f\left(\lambda x\right)] = g\left(\lambda v\right). \tag{0.9}$$

In particular, since constants are fixed by $\mathbb{S}^q$, taking $\lambda = 0$, in (0.8), yields,

$$g(0) = f(0), \tag{0.10}$$

and,

$$g^{(k)}(0) = (k!)^q f^{(k)}(0). \tag{0.11}$$

Furthermore, consistently with equation (0.2), for $q = -1$, we should have,

$$\mathbb{S}^{-1}\sum_{k=0}^{\infty} b_k v^k = \sum_{k=0}^{\infty} \mathbb{S}^{-1}[b_k v^k] = \sum_{k=0}^{\infty} \frac{b_k}{k!}x^k. \tag{0.12}$$

As an operator, $\mathbb{S}^{-1}$, then defines an inverse for the transform, $\mathbb{S}$, acting on general power series, as prescribed. The previous relations suggest that the transform, $\mathbb{S} = \mathbb{S}^1$, and its real powers may then be used as a processing or detection tool to adequately reduce or magnify a signal to a more visible or more observable ranges without altering original units or scaling factors. For instance, in situations where the original signal is weak we can use $\mathbb{S}^q$, (very small a_k coefficents, for large k) with large positive powers, q, as necessary, to render the original signal more detectable form. Similarly, for $q > 0$, $\mathbb{S}^{-q}$, the inverse operator for $\mathbb{S}^q$, can be used to reduce to manipulable size fast growing signals, with large tail ends.

By the Weierstrass approximation theorem, if a function, $f(x)$, can be expanded as a series,

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, x \in (-\epsilon,\ \epsilon), \tag{0.13}$$

then for some $\epsilon > 0$,

$$a_k = f^{\ (k)}(0)/k!, k \geq 0. \tag{0.14}$$

Hence, in this case,

$$\mathbb{S}^q[f(x)] = \sum_{k=0}^{\infty} \mathbb{S}[(f^{\ (k)}(0)/k!)x^k] = \sum_{k=0}^{\infty} (k!)^{q\ -1} f^{\ (k)}(0) v^k, \tag{0.15}$$

and in particular, for $q = 1$,

$$\mathbb{S}[f(x)] = \sum_{k=0}^{\infty} f^{\ (k)}(0) v^k = g(v). \tag{0.16}$$

Clearly, if $g_n(v)$ denotes the transform of the n^{th}formal derivative, $f^{\ (n)}(x)$, then,

$$g(0) = f(0),\ \&,\ \ g_n(0) = \mathbb{S}[f^{\ (n)}(x)]_{v=0} = n! f^{\ (n)}(0), \tag{0.17}$$

and, when defined,

$$g(\pm 1) = \sum_{k=0}^{\infty} (\pm 1)^k f^{\ (k)}(0). \tag{0.18}$$

The values, $g(\pm 1)$, may in fact be infinite as the image series may not converges except say at the singleton, $\{0\}$. The largest interval for which equations (0.11) & (0.12) hold is called the Region of Convergene, (RoC), of the series, $\sum\limits_{k=0}^{\infty} a_k x^k$. It is expected that the RoC of the image series is always a subset of the original series RoC.

However, not all functions can be written as a power series as prescribed by the Weierstrass theorem, (see for instance, p. 675 in [9], or Chap 4 in [14]). Despite its infinite differentiability everywhere, the function,

$$h(x) = \begin{cases} \exp(-1/x^2), \text{if } x \neq 0, \\ 0, \quad \text{if } x = 0, \end{cases} \tag{0.19}$$

having derivatives given by,

$$h^{(n)}(x) = Q_n(1/x)\exp(-1/x^2), \text{ with, } Q_{n+1}(t) = t^2[2tQ_n(t) - dQ_n(t)/dt], \tag{0.20}$$

does *not* satisfy the Weirstrass approximation theorem, and hence cannot be represented as a (Maclaurin) power series, and so is not admissible to the domain of the operator, $\mathbb{S}$, and its powers, as defined so far. This discrepancy hints to the extension of the transform, $\mathbb{S}$, and its powers, $\mathbb{S}^q$, to continuous counterparts (also see Sec. 2 in [5]).

1. The Sumudu, $\mathbb{S}$, and the Inverse Sumudu, $\mathbb{S}^{-1}$.

The transform, $\mathbb{S}$, of the motivational section is reminiscient of the "discrete" version of the *Sumudu,* [4]. The Sumudu transform introduced by Watugala, is bilateral in the sense its admissible functions are defined on both sides of the real line [7, 20]. Over the set of real functions,

$$A = \{f(t)/\exists\ M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}},\ \text{for } t \in (-1)^j \times [0, \infty)\}, \tag{1.1}$$

the *Sumudu* of the function, $f(t)$, is defined by,

$$G(u) = \mathbb{S}[f(t)](u) = \int_0^\infty f(ut)e^{-t}dt,\ u \in (-\tau_1, \tau_2). \tag{1.2}$$

For instance, for $a \geq 0$,the Heaviside function or unit step function, $H_a(t)$, (see for instance, p.278 [15]),

$$H_a(t) = \begin{cases} 1, \text{for } t > a, \\ 0, \text{for } t < a, \end{cases} \tag{1.3}$$

any function shift, $f(t-a)$, has for sumudu,

$$\mathbb{S}[f\,(t-a)] = \mathbb{S}[H_a(t)f(t)] = e^{\frac{-a}{u}}\mathbb{S}[f(t)], \text{ for } u > a. \tag{1.4}$$

Denoting the gamma function by, Γ, wherever, $\Gamma(\alpha+1)$ can be defined (classically for $\alpha > -1$), [13, 17],

$$\mathbb{S}\,[t^\alpha] = \int_0^\infty (ut)^\alpha\, e^{-t}dt = u^\alpha \int_0^\infty t^\alpha e^{-t}dt = \Gamma\,(\alpha+1)\,u^\alpha. \tag{1.5}$$

Computationally, we then have, $\mathbb{S}[1]=1$, $\mathbb{S}\,[at+b] = au+b$, and $\mathbb{S}[t^n/n!] = u^n$, for any integer $n \geq 0$. Hence, for $a > 0$, the sumudu, $\mathbb{S}[\exp(at)] = \mathbb{S}[\sum_0^\infty (at)^n/n!] = \sum_0^\infty (au)^n = 1/(1-au)$, for $a \in (-1/a, 1/a)$. Consequently, for any frequency, w, $1/(1+(wu)^2)$, $(wu)/(1+(wu)^2)$, are the respective *Sumudi* of $\cos(wt)$, and, $\sin(wt)$, which may be obtained through linearity and the fact, $\mathbb{S}[\cos(wt) + j\sin(wt)] = \mathbb{S}[e^{jwt}] = 1/(1-jwu) = (1+jwu)/(1+(wu)^2)$. Conversly, taking into account that (with $w = 1$), $1/(1+u^2) = \sum_0^\infty(-1)^n u^{2n}$,with u in $(-1, 1)$, and applying the operator from the previous section, $\mathbb{S}^{-1}\sum_0^\infty(-1)^n u^{2n} = \sum_0^\infty(-1)^n t^{2n}/(2n)! = \cos t$,with t in $(-\infty, \infty)$. Similarily, $\mathbb{S}^{-1}[u/(1+u^2)] = \mathbb{S}^{-1}\sum_0^\infty(-1)^n u^{2n+1} = \sum_0^\infty(-1)^n t^{2n+1}/(2n+1)! = \sin t$. (see Sumudu Pairs Tables in [6-,7, 13, 16]).

An alternative method for Sumudu computation was recenly established by Rana et al. in [16] and applied to basic functions as above, and get similar results. There, He's perturbation method, (HPM), was adapted to construct a homotopy and a power series converging to the Sumudu transform of any given function. The series terms are computed by finding derivatives rather than performing integration.

It turns out that the Inverse Sumudu is just a continuous extention of the discrete operator, $\mathbb{S}^{-1}$, introduced in the previous section by equation (0.12) . One way of establishing the Inverse Sumudu, $\mathbb{S}^{-1}$, is through connecting the Residue Theorem with Cauchy's, as was first established in [21].

Theorem 1.1: Let $G(u)$ denote the Sumudu of the function, $f(t)$, such that,

(i) the function, $xG(x)$, is a meromorphic, with singularities having, $\mathrm{Re}(x) < \beta$, *and*

(ii) there exist a circular region C with radius R, and positive constants, M and N, such that,

$$\mid G(x) \mid < M(R^N \mid x \mid)^{-1}, \tag{1.6}$$

then, modulo null functions, $f(t)$, is uniquely given by,

$$\mathbb{S}^{-1}[G(u)] = \frac{1}{2\pi j}\int_{\beta-i\infty}^{\beta+i\infty} e^{ut}\frac{G(1/u)}{u}du = \sum \text{Residues}\ [e^{ut}G(1/u)/u]\ . \quad (1.7)$$

For instance, the Inverse Sumudu of the function, $1/(1+u)$, is given by,

$$\mathbb{S}^{-1}[1/(1+u)] = \text{Residue}\ [e^{ut}/(1+u)] = \lim_{u\to\ -1}\frac{e^{ut}(1+u)}{(1+u)} = \lim_{u\to\ -1} e^{ut} = e^{-t}. \quad (1.8)$$

Furthermore, the Inverse Sumudu is related to, $\mathbb{S}^{-1}$, of the previous section in the following manner.

Corollary 1.2: The Inverse Sumudu coincides with the operator, $\mathbb{S}^{-1}$*, defined by equation (0.12), in the previous section, when restricted to powers series,* $\sum\limits_{k=0}^{\infty} b_k u^k$.

Proof: Without loss of generality, we use the already defined power series,

$f_p(u) = G(u)$, then by *Theorem 1.1*,

$$\mathbb{S}^{-1}[f_p(u)] = \mathbb{S}^{-1}[\sum_{k=0}^{\infty} u^k/[k!]^p] = \sum \text{Residues}\ [e^{ut}\sum_{k=0}^{\infty} 1/u^{k+1}[k!]^p]. \quad (1.9)$$

Therefore, as expected,

$$\mathbb{S}^{-1}[f_p(u)] = \sum_{k=0}^{\infty}\sum \text{Residues}[e^{ut}/u^{k+1}[k!]^p] = \sum_{k=0}^{\infty}\lim_{u\to\ 0}\frac{u(ut)^k/k!}{u^{k+1}[k!]^p} = \sum_{k=0}^{\infty} t^k/[k!]^{p+1} = f_{p+1}(t). \quad (1.10)$$

Various proofs of the next theorem can be found in some referenced Sumudu related papers,

Theorem 1.3: Let $G(u) = G_0(u)$ *denote the Sumudu of the function,* $f(t) = f^{(0)}(t)$*, and* $G_n(u)$ *denote the Sumudu of the n′th derivative,* $f^{(n)}(t)$*, of the function,* f*, then for* $n \geq 1$,

$$G_n(u) = u^{-n}[G(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)]. \quad (1.11)$$

Corollary 1.4: If, $G(u) = G_0(u)$, denotes the sumudu of the function, $f(t) = f^{(0)}(t)$, and $G_n(u)$, denote the sumudu of the n'th derivative, $f^{(n)}(t)$, of the function, $f(t)$, then for $n \geq 1$, and if, $G(u)$, satisfies the Weierstrass polynomial approximation theorem in an interval RoC =I, then for $u \in I$,

$$\lim_{n\to\infty} u^n G_n(u) = 0, \tag{1.12}$$

$$G(u) = \sum_{k=0}^{\infty} u^k f^{(k)}(0), \tag{1.13}$$

and,

$$G^{(k)}(0) = k! f^{(k)}(0). \tag{1.14}$$

Proof: Since, $G(u)$, satifies the Weierstrass polynomial approximation theorem, then from equation (1.11) in the previous theorem, we have,

$$\lim_{n\to\infty} u^n G_n(u) = \lim_{n\to\infty} [G(u) - \sum_{i=0}^{n-1} u^k f^{(k)}(0)] = 0. \tag{1.15}$$

This implies that when $G_n(\pm 1)$ are defined, then,

$$\lim_{n\to\infty} G_n(\pm 1) = 0. \tag{1.16}$$

At first glance, this seems to be in conflict with entries 6 & 7 of Table1, in [20], as since any derivative of sin$t$,(cos$t$), is confined into one of four possibilities, $\pm$cos$t$, $\pm \sin t$, albeit in the well known cyclical manner, $G_n(\pm 1)$, respectively generate the sequence made of infinite repititions of, $\pm(...1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2...)$ which converge nowhere, and hence violate equation (1.16). However, the resolution of this conflict lies in the fact that equation (1.16), should be considered only in the context of the Weirestrass approximation theorem, guaranteeing a specific RoC for the Sumudi. The RoC for the sumudi series version of $\sin t$ and $\cos t$ being, $(-1, 1)$,as observed above, $G_n(\pm 1)$ are undefined in the firts place. A similar argument can be said for the functions, $e^{\pm t}$, in entries 3,4 of Table1 in [20]. On the other hand, there seems to be an averaging process of cluster points for the limit in (1.16).

Corollary 1.4: For any integer, $n \geqslant 1$, the non-trivial solution to the equation, $G_n(u) = G(u)$, is given by,

$$G(u) = \frac{1}{1-u} = \mathbb{S}[e^t], u \in (-1,1). \tag{1.17}$$

Proof: For any integer, $n \geqslant 1$, from equations (1.11), we have,

$$(1-u^n)G(u) = \sum_{k=0}^{n-1} u^k f^{(k)}(0) \Leftrightarrow G(u) = \sum_{k=0}^{n-1} u^k f^{(k)}(0)/(1-u^n) \Leftrightarrow G(u) = 1/(1-u). \tag{1.18}$$

Furthermore,

$$\lim_{n\to\infty} u^n G_n(u) = \lim_{n\to\infty} \frac{u^n}{1-u} = 0 \Leftrightarrow |u| < 1. \tag{1.19}$$

One must be attentive to how the sumudu is interpreted when, t, or, u, take the values, 0, and, ∞. This should be treated in the conext of the function, $f(t)$, understudy. For suppose, $f(t)$ is defined in the interval $(-a_1, a_2)$, where a_1, a_2, may be infinite, then,

$$\lim_{u\to 0^{\mp}} G(u) = \lim_{t\to 0^{\mp}} f(t), \text{ and, } \lim_{u\to(-1)^i\tau_i} G(u) = \lim_{t\to(-1)^i a_i} f(t). \tag{1.20}$$

2. Some Sumudu Convolution Properties.

Now, we present various useful Sumudu convolution results (for a proof see for instance [3]).

Theorem 2.1: Let $M(u)$, and $N(u)$, be the respective sumudi for the functions, $f(t)$ and $g(t)$, with convolution,

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \tag{2.1}$$

then, the Sumudu of the convolution of the functions, $f(t)$, and $g(t)$, is given by,

$$\mathbb{S}[(f*g)(t)] = uM(u)N(u). \tag{2.2}$$

Furthermore, the Sumudu of the derivative of the convolution of the functions, $f(t)$ and $g(t)$,is given by,

$$\mathbb{S}[(f * g')(t)] = M(u)N(u). \tag{2.3}$$

Moreover, the Sumudu of the derivative of the convolution of f with itself, is given by,

$$\mathbb{S}[(f * f')(t)] = M^2(u). \tag{2.4}$$

Finally, the Sumudu of the anti-diderivative of the function, $f(t)$, is given by,

$$\mathbb{S}[\int_0^t f(\tau)d\tau] = uM(u). \tag{2.5}$$

For instance, since $\int_0^t \cos\tau d\tau = \sin\tau$, and since, $\mathbb{S}[\cos(t)] = 1/(1+u^2)$,then as expected we have,

$$\mathbb{S}[\sin(t)] = u\mathbb{S}[\cos(t)] = u/(1+u^2) \tag{2.6}$$

Denoting by $f^{[n]}$, the $n-$time convolution of, f , with itself, for $n \geq 0$, referring to equation (2.2), we have,

$$\mathbb{S}[f^{[n+1]}(t)] = u^n(\mathbb{S}[f(t)])^{n+1}. \tag{2.7}$$

For instance, the sumudi of the $n-$time convolutions of the expontial function yield the expressions,

$$\mathbb{S}[(e^t)^{[n+1]}] = u^n/(1-u)^{n+1}. \tag{2.8}$$

Furthermore, differentiation of the $n-$time convolutions, yields the following sumudu expression,

$$\mathbb{S}[(f^{[n+1]})'(t)] = u^n(\mathbb{S}[(f^{[n]} * f\ ')(t)]) = u^{n-1}(\mathbb{S}[f(t)])^{n+1}. \tag{2.9}$$

In particular, for $n \geq 1$, we have,

$$\mathbb{S}[(\sin(t))^{[n]} * \cos(t)] = u^{n-1}(\mathbb{S}[\sin(t)])^{n+1} = u^{n-1}(u/(1+u^2))^{n+1} = u^{2n}/(1+u^2)^{n+1}. \tag{2.10}$$

This result generalizes equation (2.20) in [3].

3. Some Sumudu Shift Properties.

Next, we establish results showing how the Sumudu transform behaves when we multiply the time function with a designated weight function.

Theorem 3.1: If $G(u)$, is the sumudu of the function, $f(t)$, then for u in the interval $(-1, 1)$ we have,

$$\mathbb{S}[e^t f(t)] = G(u/(1-u))/(1-u). \tag{3.1}$$

Proof: From equation (2.1), we have,

$$\mathbb{S}[e^t f(t)] = \int_0^\infty f(ut)\, e^{-(1-u)t} dt. \tag{3.2}$$

For u in $(-1, 1)$, using the change of variable, $w = (1-u)\,t$, we get, $dw = (1-u)dt$, and

$$\mathbb{S}\left(e^t f(t)\right) = \frac{1}{1-u} \int_0^\infty f(uw/(1-u))e^{-w} dw = \frac{G(u/(1-u))}{1-u}. \tag{3.3}$$

For instance, knowing that the sumudu of t^n is $G(u) = n!u^n$, the theorem implies that with $n \geq 0$, and u in the interval $(-1, 1)$, we have,

$$\mathbb{S}[t^n e^t] = n!u^n(1-u)^{-(1+n)}. \tag{3.4}$$

Consequently, invoking equation (2.8), we see that,

$$\mathbb{S}[t^n e^t]/\mathbb{S}[(e^t)^{[n+1]}] = n!. \tag{3.5}$$

Equation (3.5) is reminiscient of highlights of convolution quotients (distributions, see [2, 12]), where for the Heaviside function, $H = H_0$, and the Dirac $\delta-$function, we have, $H/H = H * H^{-1} = \delta$, and,

$$(H(t))^{[n+1]} = t^n/n!. \tag{3.6}$$

The inverse Sumudu being unique up to null functions by *theorem 1.1*, combining (3.5) and (3.6), we get,

$$(e^t)^{[n+1]} = e^t t^n/n! = e^t(H(t))^{[n+1]}. \tag{3.7}$$

Now we generalize further some previously established results.

Theorem 3.2: (Belgacem[2006]) If, $G(u)$, denotes the sumudu of the function, $f(t)$, and if $G^{(k)}(u)$ denotes its k'th derivative for $k \geq 0$, then for $n = 0, 1, 2...$, the sumudu of the product, $t^n f(t)$, is given by,

$$\mathbb{S}[t^n f\,(t)] = u^n \sum_{k=0}^{n} C_k^n P_{n-k}^n u^k G^{(k)}(u). \tag{3.8}$$

In particular, this theorem generalizes results for $n = 1\&\ 2$,in [1],

$$\mathbb{S}[tf\,(t)] = u[G(u) + uG\,'(u)], \tag{3.9}$$

and,

$$\mathbb{S}[t^2 f\,(t)] = u^2[2G(u) + 4uG\,'(u) + u^2 G\,''(u)]. \tag{3.10}$$

Corollary 3.3: If, $G(u)$, denotes the sumudu of the function, $f(t)$, $G_m(u)$, denotes the sumudu of the m'th derivative of, $f^{\,(m)}(t)$, and $G_m^{(k)}(u)$ denotes its k'th derivative with respect to u, then the sumudu of the product, $t^n f^{\,(m)}(t)$, is given by,

$$\mathbb{S}[t^n f^{\,(m)}\,(t)] = u^n \sum_{k=0}^{n} C_k^n P_{n-k}^n u^k G_m^{(k)}(u),\ m,\ n = 0, 1, 2... \tag{3.11}$$

For instance, we have,

$$\mathbb{S}[t^2 f\ '(t)] = u^2[2G_1(u) + 4uG_1'(u) + u^2 G_1''(u)], \tag{3.12}$$

$$\mathbb{S}[t^3 f\ '(t)] = u^3[G_1(u) + 6uG_1'(u) + 9u^2 G_1''(u) + u^3 G_1'''(u), \tag{3.13}$$

and,

$$\mathbb{S}[t^4 f\ ''(t)] = u^4[24G_2(u) + 96uG_2'(u) + 72u^2 G_2''(u) + 16u^3 G_2'''(u) + u^4 G_2''''(u). \tag{3.14}$$

Not withstanding some difference in notation, equations (3.12-3.14), are to be verified against Corollary 5.4 in [6]. In fact, theroem 3.2 and Corollary 3.3 are equivalent since each one implies the other. However, the expression in the latter can afford to be improved by using theorem 4.3 and its corallaries in [5] to have formulation (3.11), be written more explicitely by expanding the term, $G_m^{(k)}(u)$. When this is properly done, a testing point is to verify that, when $n = m$, the sumudu maintains unit and scale consistency, already proved in [6], by direct integrative means,

$$\mathbb{S}[t^n f^{(n)}(t)] = u^n G_n(u). \tag{3.15}$$

4. Some Sumudu Applications to Bessel Equations and Functions.

Bessel functions long history dates back to Bernoulli in 1732, who disovered the oscillatory behavior of heavy chains. They were then used by Euler in 1764 to describe circular drumhead vibrations automatically admitting them to the set of cylindrical functions. They came to bear the name of F. W. Bessel (1784-1846), as solutions to the differential equations describing dynamics of planetary motion, [11]. In the Sumudu context, Bessel functions were first treated among other special functions by Asiru [1]. Bessel functions of the first kind of order n, are given by,

$$J_{\pm n}(t) = \sum_{k=0}^{\infty}(-1)^k (t/2)^{2k\pm n}/k!\Gamma(k+1\pm n), \quad n = 0, 1, 2, ... \tag{4.1}$$

resolve the Bessel equations,

$$t^2y''(t) + ty'(t) + (t^2 - n^2)y = 0, \;\; n = 0, 1, 2, ... \tag{4.2}$$

Using the discrete Sumudu definition we get,

$$\mathbb{S}[J_{\pm n}(t)] = \sum_{k=0}^{\infty}(-1)^k(u/2)^{2k\pm n}(2k \pm n)!/k!\Gamma(k+1 \pm n), n = 0, 1, 2, ... \tag{4.3}$$

according to the shift theorems in the previous section, must then solve the *"sumuded"* equations,

$$(u^2 + u^4)G''(u) + (u + 4u^3)G'(u) + (2u^2 - n^2)G(u) = 0, n = 0, 1, 2, ... \tag{4.4}$$

Since,

$$J_0(t) = \sum_{k=0}^{\infty}(-1)^k[t^k/2^k k!]^2, \tag{4.5}$$

with u in $(-1, 1)$,

$$\mathbb{S}[J_0(t)] = \sum_{k=0}^{\infty}(-1)^k[2k!/(k!)^2](u/2)^{2k} = \sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}(u/2)^{2k} = 1/\sqrt{1+u^2}, \tag{4.6}$$

is then a solution to the seemingly more cumbersome transformed Bessel equation of order zero,

$$(u^2 + u^4)G''(u) + (u + 4u^3)G'(u) + 2u^2G(u) = 0. \tag{4.7}$$

Since, $J_0(2\sqrt{t})] = \sum_{k=0}^{\infty}(-1)^k t^k/(k!)^2$, then,

$$\mathbb{S}[J_0(2\sqrt{ut})] = \mathbb{S}[\sum_{k=0}^{\infty}(-1)^k u^k t^k/(k!)^2] = \sum_{k=0}^{\infty}(-1)^k u^{2k}/k! = \exp(-u^2), \tag{4.8}$$

and,

$$\mathbb{S}[J_0(2\sqrt{t})] = \mathbb{S}[\sum_{k=0}^{\infty}(-1)^k t^k/(k!)^2] = \sum_{k=0}^{\infty}(-1)^k u^k/k! = \exp(-u), \tag{4.9}$$

which is consistent with entry 36 in Table A.1 in [7].

Applying the Sumudu once more, to the image in equation (4.9), we get,

$$\mathbb{S}^2[J_0(2\sqrt{t})] = \mathbb{S}[\exp(-u)] = 1/(1+w),\ w \in (-1,1). \tag{4.10}$$

Referring back to equation (0.1), using the following path, $(x \to t \to u \to w)$, mapwise, we have,

$$\mathbb{S}^{-1}[J_0(2\sqrt{t})] = \mathbb{S}^{-2}[\exp(-u)] = \sum_{k=0}^{\infty}(-1)^k x^k/(k!)^3 = f_3(-x), \tag{4.11}$$

and,

$$\mathbb{S}^{-1}[J_0(t)] = \mathbb{S}^{-2}[1/\sqrt{1+u^2}] = \sum_{k=0}^{\infty}(-1)^k x^{2k}/[(2k)!(k!)^2 2^{2k}], \tag{4.12}$$

then must solve the integro-differential equation (*Inverse* Sumdu transform of the Bessel Equation of zero order),

$$x^2 Y''(x) + xY'(x) + \int_0^x \int_0^x Y(\xi)(d\xi)^2 = 0, \tag{4.13}$$

where, $\mathbb{S}^{-1}[y(t)] = Y(x)$. Setting, $Y(x) = Z''(x)$, equation (4.13) above becomes,

$$x^2 Z'''' + xZ''' + Z = 0, \tag{4.14}$$

with power series solution,

$$Z(x) = x^2 \sum_{k=0}^{\infty}(-1)^k x^{2k}/[(2k+1)(k+1)(2k)!(k!)^2 2^{2k+1}]. \tag{4.15}$$

Noteworythy is that not only the Sumudu transform or postive integer powers can be used to transform differential equations, but that at times the inverse sumudu or negative integer powers can be even more useful in this regard. It remains to show whether taking fractional powers, q, can also be as useful, [13]. Future attempts however must first define the analogus integral formulations to (1.2), for $\mathbb{S}^q$, when q is real. In doing so the Bessel-Sumudu connetion will turn out to be useful, as is shown below.

5. The Bilateral Laplace-Sumudu Duality (BLSD).

For a real function, $f(t)$, the ordinary Laplace transform, is given by,

$$F(s) = \mathbb{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)dt, \mathrm{Re}\, s \geq 0. \tag{5.1}$$

The Laplace transform pairs table in [18], page 162, entry (32.19), shows that,

$$\mathbb{L}^{-1}[F(1/s)/s](t) = \int_0^\infty J_0(2\sqrt{vt}) f\,(v)\,dv. \tag{5.2}$$

On the other hand, the bilateral Laplace transform of the function, $f(t)$, is given by,

$$\mathbb{B}[f(t)](s) = \int_{-\infty}^{+\infty} f(t)e^{-st}dt = \mathbb{L}[f(t)](s) + \mathbb{L}[f(-t)](-s). \tag{5.3}$$

Up to date most writings if not all regarding the Sumudu, refer at some point or another to a form of a reciproity between the Sumudu and the Laplace transform. The Sumudu itself being defined for u, and t,changing sign, this is best stated in the context of the Bilateral Laplace-Sumudu Duality (**BLSD)**, (see [3]),

$$\mathbb{S}[f(t))](u) = \frac{1}{u}\mathbb{B}[f(t)](1/u), \text{ and, } \mathbb{B}[f(t))](s) = \frac{1}{s}\mathbb{S}[f(t)](1/s),\ u, s \neq 0. \tag{5.4}$$

Since by definition the Sumudu functional value at $u = 0$, is fixed to the antecedant's (also see equation (1.20)),

$$G(0) = \mathbb{S}[f(t))](0) = f(0), \tag{5.5}$$

for pragmatic simplicity of expression, we may split the Sumudu, $\mathbb{S}$, to "positive" and "negative" components, $\mathbb{S}^+$, and, $\mathbb{S}^-$, each acting on the correspondingly signed side of the real axis,

$$\mathbb{S}[f(t)](u) = \begin{cases} \mathbb{S}^+[f(t)](u) = \mathbb{S}[H(t)f(t)](u), & \text{for } \ t, \ u > 0, \\ \mathbb{S}^-[f(t)](u) = \mathbb{S}[H(-t)f(t)](u), & \text{for } \ t, \ u < 0. \end{cases} \tag{5.6}$$

This means that the **BLSD** can also be broken apart. In particular, we can recover the duality between ordinary Laplace transform and, $\mathbb{S}^+$, (**LSD**, [6-, 7]),

$$\mathbb{S}^+[f(t))](u) = \frac{1}{u}\mathbb{L}[f(t)](1/u) = F(1/u)/u, \text{ and, } \mathbb{L}[f(t))](s) = \frac{1}{s}\mathbb{S}^+[f(t)](1/s), \ u, s > 0. \tag{5.7}$$

Combining equations (5.2), and (5.7), for s positive real, we have an alternative characterization for, $\mathbb{S}^+$.

Theorem 5.1: *The Positive Sumudu operator, $\mathbb{S}^+$, of the function, $f(t)$, can be expressed in terms of the Laplace transform and the zeroth order Bessel function, J_0, as,*

$$\mathbb{S}^+[f(t)](u) = \frac{1}{u}\int_0^\infty e^{-t/u} f\,(t)\,dt = \mathbb{L}[\int_0^\infty J_0(2\sqrt{vt}) f\,(v)\,dv](s = u), u > 0. \tag{5.8}$$

For instance, since, the Heaviside function is fixed by the Sumudu

$$\mathbb{S}^+[H(t))](u) = H(u) = 1, \text{ for } u > 0, \tag{5.9}$$

by equation (5.1), we have,

$$\mathbb{L}^{-1}[1](t) = \int_0^\infty J_0(2\sqrt{vt})dv, \text{ for } t > 0. \tag{5.10}$$

The previous theorem and similar results involving higher order Bessel functions can most probably be extended, with some care to both components of the Sumudu, $\mathbb{S}$, and the Bilateral Laplace transform, $\mathbb{B}$, through the **BLSD**.

6. Epilogue: In Defense of the Sumudu.

On one hand the **BLSD** helps establish useful paralleles between the Sumudu and Laplace transforms. These conversions helped generate and check the Transform pairs tables in [6-, 7, 13]. On the other hand, it instigates concerns over what possible advantages can the Sumudu bring to the mathematical community over the Laplace transform. Being mathematical tools, one may think of various coordinate systems Euclidian vs cylindrical or spherical, Eularian vs Lagrangian systems viewpoints, Stratonovich vs Ito stochastic calculi, and even the Maple vs Mathematica symbolic algebra packages, and there is room and need for all. For at first glance, it may just be a matter of taste, or simply familiarity. The Laplace transform claiming a history advantage nearing three centuries, the majority of the engineering, mathematical, and physical sciences communities at large, have yet to be exposed to this two-decade old (young) transform. However, we believe the issue goes deeper than mere convenience.

The mathematical litterature abounds with theories with more than one name attached to them, or with multiple dualities. If the Sumudu can advance some mathematical theory further, fasten some computation techniques, provide easier (user friendly) tools for solved problems, or help solve new ones, then its purpose for existence would be more than justified. There are clear indications that the Sumudu can do all of the above, [1, 4, 13, 17, 20-23]. In fact, with almost every new paper on the subject, there is growing evidence, for the Sumudu as a stand alone viable tool rivaling usally used transforms in typical problem solving, or serving a separate mathematical purpose all together.

For instance, in [16], the authors succefuly used He's Perturbation Method (HPM) to construct a homotopy that helps compute the Sumudu transform using differentiation instead of integration which is expected to foster ramified theoretical advances, and many more applications with functions that are not integration prone. The authors of [19] combined the Sumudu transform to analytically solve the two dimensional neutron transport equation. In [10] Maxwell's equations were Sumudu treated, and the Sumudu particular reciprocity property neutralized the need for inversion since the Electrical solution profiles characteristics could be read off its transform.

To defend the Sumudu, we first recall that such legitimate concerns were in fact brought up more than a decade ago, and were in part but eloquently adressed in [8]. We say that for pedagogical purposes, the Sumudu seems a bit easier to handle, due to its discrete version simplicity, and its powers, $\mathbb{S}^q$, formulation introduced at the beginning of this paper. Furthermore, the discrete Sumudu various fractional powers, and the HPM method in [16] may lend more strength to Sumudu use. Moreover, the Sumudu convolution, *Theorem 2.1,* and possible generalizations can probably help invert some nonlinear differential equations into into more resolvable formats, such as convolution, integral or integro-differential ones. In fact, as was indicated in *section 4* above, and expressed in [6], many differential equations may become more tractable when they are inverse "*sumuded*", than when forwardly transformed.

This flexibility seems not to exist with other transforms because they do not preserve units, and the Sumudu does, which makes it appear a bit more "natural". The Sumudu has many other useful preserving properties related to scaling, and averages...etc. This suggests that it may be preferable to use the Sumudu when the variable, $u = 1/s$, is real, while the more familiar transforms such as Laplace and Fourier can be used more when, $s = 1/u$, is complex or pure imaginary. This is why we think it may be feasible to obtain a real inversion formula for the Sumudu that may be useful, but that does not use complex theory as in *Theorem 1.1.* We believe, the Sumudu has already made strides, and has a firm mathematical future almost surely, especially in physical and engineering applications. The Sumudu transform widespread use, if any, will reinforce the truth of this stance, or negate it in the contrary, and time will tell!.

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Table 1. Sumudu Properties

Formula	Comment
$G(u) = \mathbb{S}(f(t)) = \int\limits_0^\infty f(ut)\, e^{-t} dt, \quad -\tau_1 < u < \tau_2$	Sumudu Definition ($f \in A$)
$G(u) = \frac{F\left(\frac{1}{u}\right)}{u}, \; \& \; F(s) = \frac{G\left(\frac{1}{s}\right)}{s}$	**BLSD**
$\mathbb{S}[af(t) + bg(t)] = a\mathbb{S}[f(t)] + b\mathbb{S}[g(t)]$	Sumudu Linearity
$\mathbb{S} \int\limits_0^t f(\tau)\, d\tau = uG(u)$	Sumudu of function antiderivative
$G_n(u) = \mathbb{S}\left[f^{(n)}(t)\right] = \frac{G(u)}{u^n} - \frac{f(0)}{u^n} - \ldots - \frac{f^{(n-1)}(0)}{u}$	Sumudu of function derivative
$\mathbb{S}[H(t-a)f(t)](u) = e^{-\frac{a}{u}} G(u)$	Heaviside Shift Theorem
$\mathbb{S}[e^{at} f(t)](u) = \frac{1}{1-au} G\left(\frac{u}{1-au}\right)$	Exponetial Shift Theorem
$\mathbb{S}[t^n f(t)](u) = u^n \sum\limits_{k=0}^{n} C_k^n P_{n-k}^n u^k G^{(k)}(u).$	Time Powers Shift Theorem
$\mathbb{S}[(f * g)(t)](u) = u\mathbb{S}[f(t)](u)\mathbb{S}[g(t)](u)$	Sumudu Convolution Theorem

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