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Computational Frameworks for Image Enhancement

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Abstract

In this paper we present two numerical techniques which can be used both in image debluring and impulsive noise problems. Our methods are based on the total variation and Shannon's entropy measure taking into account convexity. Numerical experiments indicate great potential and promising results for future extensions.

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1 Introduction

Constrained image enhancement is an area of ongoing research ([2]). Due to imperfections in the image mediums, the observed image often represents a degraded version of the original scene. The degradations may have many causes, but two types of degradations are often dominant: blurring and noise. Blurring can be caused by relative motion between the camera and the original scene, or by an optical system which is out of focus. Linear motion blur, uniform out-of-focus blur, atmospheric turbulence blur, scatter blur are some of the popular blur types. In addition to blurring effects, the observed image may also be corrupted by noises. In imaging, the term noise refers to random fluctuations in intensity values that occur during image capture, transmission, or processing, and that may distort the information given by the image.

In many imaging systems an ideal image which is subject to both blur and additive noise takes the form

$$\Psi_0 = H\Psi + \eta, \tag{1.1}$$

where Ψ_0 and Ψ are the observed and the original images, respectively, H is blur matrix function and η is the noise constraint. According to the noisy blurred image model (1.1), the purpose of image restoration (or in particular, image debluring in our case) can now be specified as the computation of an estimate $\widehat{\Psi}$ of the original image Ψ when Ψ_0 is observed, H is known and some statistical knowledge of the noise, η, is available. In this paper we assume that the noise, η, is Gaussian with zero mean and variance of noise is given by σ^2. This leads to solving the following noise and blur constrained optimization problem

$$\begin{array}{ll} \min_{\Psi} & F(\Psi) \\ \text{s.t.} & ||H\,\Psi - \Psi_0||^2 = \sigma^2, \end{array} \tag{1.2}$$

where the norm $||\cdot||$ is used in $L^2(\Omega)$ norm sense and F is given functional which is often a criterion of smoothness of the reconstructed image. Using Lagrange multipliers theorem, the minimization of (1.2) is given by

$$\widehat{\Psi} \;=\; \text{argmin}_{\Psi} \left\{ F(\Psi) + \frac{\lambda}{2}||H\,\Psi - \Psi_0||^2_{L^2(\Omega)} - \sigma^2 \right\} \tag{1.3}$$

where λ is positive constant satisfying $||H\,\Psi - \Psi_0||^2_{L^2(\Omega)} = \sigma^2$. In general the functional $F(\Psi)$ is chosen as

$$F(\Psi) \;=\; \int_{\Omega} \Phi(|\nabla\Psi|)d\Omega, \tag{1.4}$$

where $\Phi : \mathbf{R}_+ \to \mathbf{R}$ is given smooth function called variational integrand or Lagrangian. The Dirichlet integral $\mathcal{D}(\Psi) = \frac{1}{2}\int_{\Omega} |\nabla\Psi|^2 d\Omega$ and the total variation integral $TV(\Psi) = \int_{\Omega} |\nabla\Psi| d\Omega$ are two of the classical integrals used in image denoising. The TV method incorporates the fact that discontinuities are present in the original image Ψ. The TV method has been used in image denoising models successfully and extensively. In this paper we use the TV model and entropy method for debluring of noisy blurred images.

The optimization problem (1.3) can be written as

$$\begin{aligned} \widehat{\Psi} \;&=\; \text{argmin}_{\Psi\in\mathbf{X}} \left\{ F(\Psi) + \frac{\lambda}{2}||H\,\Psi - \Psi_0||^2_{L^2(\Omega)} - \sigma^2 \right\}, \qquad (1.5) \\ &=\; \text{argmin}_{\Psi\in\mathbf{X}} \int_{\Omega} \left(\Phi(|\nabla\Psi|) + \frac{\lambda}{2}|H\Psi - \Psi_0|^2 - \sigma^2 \right) d\Omega \end{aligned}$$

where $\mathbf{X}$ is an appropriate image space of smooth functions like $C^1(\overline{\Omega})$, or the space $BV(\Omega)$ of image functions with bounded variations, or the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$.

A problem is well-posed when: (i) The solution exists for any data, (ii) the solution is unique, (iii) the solution depends continuously on the data, i.e. it is stable. Whenever any of the requirements above are not satisfied, the problem is said to be ill-posed. In order to show that our minimization problem (1.5) is well-posed we present the following theorem:

Theorem 1.1 *[4] Let the image space* $\mathbf{X}$ *be a reflexive Banach space, and let* F *be (i) weakly lower semicontinuous, i.e., if for any sequence* (Ψ^n) *in* $\mathbf{X}$ *converging weakly to* Ψ*, we have* $F(\Psi) \leq \liminf_{n\to\infty} F(\Psi^n)$*. (ii)* $F(\Psi) \to \infty$ *as* $||\Psi|| \to \infty$*. (Coercive). Then the functional* $\mathcal{L}$ *is bounded from below and possesses a minimizer, i.e., there exists* $\widehat{\Psi} \in \mathbf{X}$ *such that* $\mathcal{L}(\widehat{\Psi}) = \inf_{\mathbf{X}} \mathcal{L}$*. Moreover, if* F *is convex and* $\lambda > 0$*, then the optimization problem (1.5) has a unique solution, and it is stable.*

Using the fundamental lemma of the calculus of variations [1], the Euler-Lagrange equation corresponding to (1.5) is given by

$$-\nabla \cdot \left(\frac{F'(|\nabla\Psi|)}{|\nabla\Psi|} \nabla\Psi \right) + \lambda H^T (H\Psi - \Psi_0) \;=\; 0, \;\; \text{in}\,\Omega, \tag{1.6}$$

with homogeneous Neumann boundary conditions where H^T denotes the transpose of the matrix H. The original image, Ψ, must satisfy this equation. Using the Euler-Lagrange variational principle, the minimizer of (1.5) can be interpreted as the steady state solution of

$$\Psi_t \;=\; \nabla \cdot \left(\frac{F'(|\nabla\Psi|)}{|\nabla\Psi|} \nabla\Psi \right) - \lambda H^T (H\Psi - \Psi_0) \;\text{in}\,\Omega \times \mathbf{R}_+, \tag{1.7}$$

with homogeneous Neumann boundary conditions. When this nonlinear elliptic PDE reaches to its steady state, $\Psi_t = 0$, which gives us (1.6).

Next we present and numerically solve our first model which is based on TV method for debluring of noisy blurred images.

2 TV model for Image Debluring Problem

In this section we choose the functional $F(\Psi)$ as TV integral variation. In other words, we want to find the minimizer of

$$\int_\Omega \left(|\nabla\Psi| + \frac{\lambda}{2} |H\Psi - \Psi_0|^2 - \sigma^2 \right) d\Omega \tag{2.8}$$

for a given positive Lagrange multipliers, λ.

Proposition 2.1 *Let $\lambda > 0$. The minimization problem*

$$\widehat{\Psi} \;=\; arg\,min_{\Psi\in H^1(\Omega)} \int_\Omega \left(|\nabla\Psi| + \frac{\lambda}{2}|H\Psi - \Psi_0|^2 - \sigma^2 \right) d\Omega$$

has a unique solution provided that $|\Psi| \geq 1$.
Proof: It is clear that $F(\Psi) = \int_\Omega |\nabla\Psi|$ satisfies the conditions of Theorem 1.1. Thus, proof follows directly from Theorem 1.1 for $|\Psi| \geq 1$.

Now using Euler-Lagrange variational principle (1.7) we have

$$\Psi_t \;=\; \nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|}\right) - \lambda H^T(H\Psi - \Psi_0) \tag{2.9}$$

To overcome the numerical difficulties, we can apply different strategies inhere such as considering $|\nabla\Psi|$ as $|\nabla\Psi|_\varepsilon \;=\; \sqrt{|\nabla\Psi|^2 + \varepsilon}$ or we can multiply the equation (1.6) with $|\nabla\Psi|$. We do the latter and the resulting equation is given as

$$\Psi_t \;=\; |\nabla\Psi|\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|}\right) - |\nabla\Psi|\lambda H^T(H\Psi - \Psi_0). \tag{2.10}$$

Since $|\nabla\Psi| \geq 0$, this step does not change the solution of the equation (1.6). Now we numerically solve the equation (2.10). In order to achieve this we discretize $|\nabla\Psi|\,\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|}\right)$ and $-|\nabla\Psi|\,\lambda H^T(H\Psi - \Psi_0)$, separately. Let Ψ^n_{ij} be the approximation to the value $\Psi(x_i, y_j, t_n)$, where $x_i = i\Delta x$, $y_j = k\Delta y$, $t_n = n\Delta t$, where Δx, Δy, Δt, spatial and time step sizes, respectively. Let us define the quantity

$$w^n_{ij} \;=\; \lambda H^T(H\Psi^n_{ij} - \Psi_0).$$

Then, a numerical solution of (2.10) can be given as

$$\frac{\Psi^{n+1}_{ij} - \Psi^n_{ij}}{\Delta t} \;=\; \mu^n_{ij} + \Lambda^n_{ij} w^n_{ij},$$

where

$$\begin{aligned}
\mu^n_{ij} &= \frac{(\theta_{xx})_{ij}\,((\theta_y)_{ij})^2 - 4\,(\theta_{xy})_{ij}\,(\theta_x)_{ij}\,(\theta_y)_{ij} + (\theta_{yy})_{ij}\,((\theta_x)_{ij})^2}{((\theta_x)_{ij})^2 \,+\, ((\theta_y)_{ij})^2 \,+\, \varepsilon} \\
(\theta_{xx})_{ij} &= \frac{\Psi^n_{i+1,j} - 2\,\Psi^n_{ij} \,+\, \Psi^n_{i-1,j}}{2\,\Delta x^2} \\
(\theta_{xy})_{ij} &= \frac{\Psi^n_{i+1,j+1} - \Psi^n_{i-1,j+1} \,-\, \Psi^n_{i+1,j-1} + \Psi^n_{i-1,j-1}}{2\,\Delta x\,\Delta y}
\end{aligned}$$

etc. and Λ^n_{ij} is any discretization of $|\nabla\Psi|$ and ε is a positive regularization term to guarantee that the denominators are not identically zero. In the next section we present an entropy based image debluring method for noisy blurred images.

3 Entropy Model for Image Debluring

The maximum entropy method ([5]) is an important principle for modeling the prior probability $p(\Psi)$ of an unknown image Ψ. It can be described as follows:

$$\begin{aligned} \min_\Psi \quad & \int p(\Psi)\,\log p(\Psi) d\Psi, \\ s.t. \quad & \int p(\Psi) d\Psi = 1, \\ \mu_r \quad = & \int m_r(\Psi) p(\Psi) d\Psi,\ r = 1\ldots, n, \end{aligned}$$

where $p(\Psi)$ is the prior probability and $m_r(\Psi)$ are moments of some known functions for each $r = 1, \ldots, n$. Using the Lagrange multiplier's theorem, the solution of this problem is given by

$$p(\Psi) \;=\; \frac{1}{Z} \exp\left\{ -\sum_{r=1}^{n} \lambda_r\, m_r(\Psi) \right\},$$

where λ_r's are the Lagrange multipliers, and Z is the partition function.

We define the variational integral (1.4) as

$$F(\Psi) \;=\; \int_\Omega |\nabla\Psi| \log|\nabla\Psi| d\Omega, \tag{3.11}$$

where $\Phi(\Psi) = \Psi\log(\Psi)$, $\Psi \geq 0$. Note that $-\Phi(\Psi) \to 0$ as $\Psi \to 0$. It follows from the inequality,

$$\Psi\log(\Psi) \leq \frac{\Psi^2}{2},$$

that

$$|F(\Psi)| \leq \int_\Omega |\nabla\Psi|^2 d\Omega \leq ||\Psi||^2_{H^1(\Omega)} < \infty,\ \text{for every}\ \Psi \in H^1(\Omega).$$

Thus the variational integral given in (3.11) as $F(\Psi) : H^1(\Omega) \to \mathbf{R}$ is well defined. It is clear that the Lagrangian $\Psi\log(\Psi)$ is strictly convex and coercive, i.e., $\Psi\log(\Psi) \to \infty$ as $|\Psi| \to \infty$. The following proposition results from Theorem 1.1.

Proposition 3.1 *Let $\lambda > 0$. The minimization problem*

$$\widehat{\Psi} \;=\; arg\,min_{\Psi\in H^1(\Omega)} \int_\Omega \left(|\nabla\Psi| \log|\nabla\Psi| + \frac{\lambda}{2}|H\Psi - \Psi_0|^2 - \sigma^2 \right) d\Omega$$

has a unique solution provided that $|\Psi| \geq 1$. Using the Euler-Lagrange variational principle (1.7) it follows that

$$\Psi_t \;=\; \nabla\cdot\left(\frac{1+\log|\Psi|}{|\Psi|}\nabla\Psi \right) - \lambda H^T(H\Psi - \Psi_0),\ \text{in}\ \Omega\times\mathbf{R}_+, \tag{3.12}$$

with homogeneous Neumann boundary conditions. Because the numerical implementation for (3.12) is similar to the one that we applied for (2.10), we skip the details of the numerical method for this case.

4 A Related Work

In recent years variational methods and PDE based methods have been applied to a variety of problems including image classification, image segmentation, image denoising, and image restoration. In this paper we focused on the latter. A large number of PDE based methods have particularly proposed to tackle the problem of image filtering with a good preservation of the edges. Figure 1 contains application of some of these techniques.

PDEs or gradient-descent flows are generally a result of variational problems using the Euler-Lagrange principle [1]. Next we present a method based on the solution of the Euler-Lagrange equations as our method does. G. Aubert and P. Kornprobst writes [3] our optimization problem (1.5) in a different form as

$$\inf_{\Psi \in \mathbf{X}} \frac{1}{2} \int_{\Omega} |H\Psi - \Psi_0|^2 d\Omega \, + \, \lambda \int_{\Omega} \Phi(|\nabla \Psi|) d\Omega,$$

where

$$\mathbf{X} := \{\Psi \in L^2(\Omega) : |\nabla|\Psi \in L^1(\Omega)^2\}.$$

They worked for both case of Φ, (convex and non-convex):

$$\begin{aligned} \Phi(\Psi) &= 2\sqrt{1+\Psi^2} - 2 \quad \text{(convex)}, \\ \Phi(\Psi) &= \log(1+\Psi^2) \quad \text{(non-convex)}, \\ \Phi(\Psi) &= \frac{\Psi^2}{1+\Psi^2} \quad \text{(convex)}. \end{aligned}$$

Let us note that our method is different from [3]: We work for different functions in a different optimal control problem setting and present a different numerical scheme from the one presented in [3].

5 Experiments

In this paper we have modeled a noisy blurred image as

$$\Psi_0 \, = \, H\Psi + \eta,$$

where the blur matrix fucntion is usually chosen as convolution operator and mostly known as point spread function (PSF). Noise is an important limitation in the identification and restoration of images. The amount of noise present in an observed image is given by the (blurred-) signal-to-noise ration (SNR):

$$SNR \, = \, 10 \log_{10} \left(\frac{\text{variance of the blurred image}}{\text{variance of the noise}} \right) \text{ (dB)}.$$

Figure 2 illustrates original, noisy, and noisy blurred images from left to right, respectively, with additive Gaussian noise.

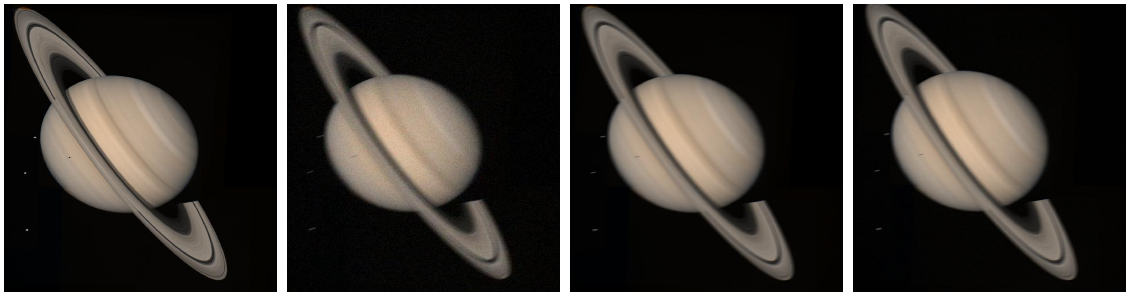

Figure 1: Original noise-blurred and restored images by TV, entropy models, left to right.

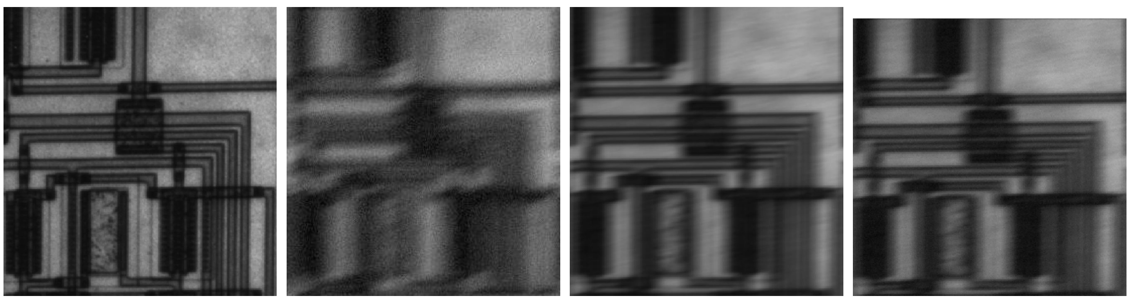

Figure 2: Original noisy and restored images by TV, entropy models, left to right.

Example 1: In this example we define the observed image $\Psi_0(x, y)$ on a square $\Omega = \{(x, y) : 0 \leq x, y \leq 1\}$ with $\Psi_0(x, y)$ taking on discrete values between 0 and 255. We choose H as a particular convolution operator as

$$H(x, y, \sigma) = \frac{1}{4\pi\sigma}\, e^{-(x^2+y^2)/(4\sigma)}$$

with a Gaussian of variance $\sigma > 0$, and $SNR = 2.0$. Figure 3 contains an illustration of resulting images obtained by applying our methods.

Example 2: In this example we choose H as the identity matrix and apply our methods for $SNR = 1.0$. Figure 4 shows the resulting images.

6 Conclusion

In this paper we present two computational methods which are based on the total variation and entropy theory for the image debluring problem. Obtained results show that our method is quite efficient for debluring noisy blurred images. Our method is general in the sense that it may be applied not only to image debluring problem but also image denoising and image identification problems.

There is an easy connection between image denoising and image debluring problems. For instance, writing the blur matrix H as the identity matrix in

(1.1), we can express the image debluring problem as image denoising problem. There is also a significant connection between image debluring and image identification problem. We can use the same model (1.1) for the image identification problem but the image identification problem we do not have the entire knowledge about H, thus, the image identification problem focuses on estimating H and the parameters of the statistical models for Ψ and η, from the observed image Ψ_0. One of our future goals is to use different variational integral functionals and apply some algebraic methods such as the least squares method for the image debluring and identification problems.

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