Applied Mathematical Sciences, Vol. 4, 2010, no. 32, 1573 - 1581

Axi-Symmetric Deformation due to Expanding Surface Loads in a Fluid Saturated Porous Medium

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Abstract

The general solution to the field equations of fluid saturated incompressible porous medium for two dimensional axi-symmetric problem is obtained due to time harmonic ring and disc loads by applying Hankel transform. These loads suddenly emanate from a point on the surface and expand radially at constant rate. These loads are chosen so that they exert a constant force on the surface as they expand. The transformed solution is inverted numerically by using a numerical inversion technique to invert the integral transform. Effect of porosity is obtained on the components of displacement and stress and depicted graphically.

Mathematics Subject Classification: 74E05, 74F10, 74J10

Keywords: Ring load, disc load, pore pressure, Hankel transform, axi-symmetric, frequency domain.

1. Introduction

Porous media theories play an important role in many branches of engineering including material science, chemical engineering, biomechanics and other such fields of engineering. Representation of a fluid saturated porous medium as a single phase material has been virtually discarded. The material with pore spaces such as concrete can be treated easily because all concrete ingredients have the same motion if the concrete body is deformed. However the situation is more complicated if the pores are filled with liquid. In this case the solid and liquid phases have different motions. Due to these different motions, the different material properties and the complicated geometry of pore structures, the mechanical behavior of a fluid saturated porous medium becomes more difficult. So researchers from time to time, have tried to overcome this difficulty and we se many porous media theories in the literature. A brief historical background of these theories is given by de Boer [2,3].

As far as modern era is concerned Biot [1] proposed a general theory of three-dimensional deformation of fluid saturated porous salts. Biot theory is based on the assumption of compressible constituents and till recently, some of his results have been taken as standard references. Another interesting theory is given by Bowen [9], de Boer and Ehlers [4] in which all the constituents of a porous medium are assumed to be incompressible. The fluid saturated porous material is modelled as a two phase system composed of an incompressible solid phase and incompressible fluid phase. Kumar and Hundal [6-7] studied some problems of deformation in fluid saturated incompressible porous media. However, no attempt has been made to study axi-symmetric deformation in fluid saturated incompressible porous media.

In the present paper, we obtained the components displacement, stress and pore pressure due to loads that suddenly emanates from a point on the surface and expands radially at constant rate in frequency domain in fluid saturated incompressible porous medium due to axial-symmetry.

2. Basic equations

Following de Boer and Ehlers [4], the equations governing the motion of an incompressible porous medium saturated with non-viscous fluid in the absence of body forces are

$$\nabla . (\eta^S \, \dot{\boldsymbol{u}}_S + \eta^F \, \dot{\boldsymbol{u}}_F) = 0, \tag{1}$$

$$(\lambda^S + \mu^S) \, \nabla(\nabla . \boldsymbol{u}_S) + \mu^S \nabla^2 \boldsymbol{u}_S - \eta^S \nabla \mathrm{p} - \rho^S \ddot{\boldsymbol{u}}_S + \mathrm{s}_V (\dot{\boldsymbol{u}}_F - \dot{\boldsymbol{u}}_S) = 0, \tag{2}$$

$$\eta^F \, \nabla \mathrm{p} + \rho^F \, \ddot{\boldsymbol{u}}_F + \mathrm{s}_V \, (\dot{\boldsymbol{u}}_F - \dot{\boldsymbol{u}}_S) = 0, \tag{3}$$

$$\boldsymbol{T}_E^{\,S} = 2 \mu^S \, \boldsymbol{E}_S + \lambda^S \, (\mathbf{E}_S . \mathbf{I})\mathbf{I}, \tag{4}$$

$$\boldsymbol{E}_S = \frac{1}{2} \, (\mathrm{grad} \, \boldsymbol{u}_S + \mathrm{grad}^T \boldsymbol{u}_S) \,, \tag{5}$$

where symbols have their usual meaning.

3. Formulation of the Problem

We consider a homogenous fluid saturated incompressible porous medium whose boundaries are parallel to the plane z = 0 in the cylindrical polar coordinate system(r, θ, z). We consider a two- dimensional axi-symmetric problem with symmetry about z axis, so that all the quantities remain independent of θ and $\frac{\partial}{\partial\theta} = 0$.

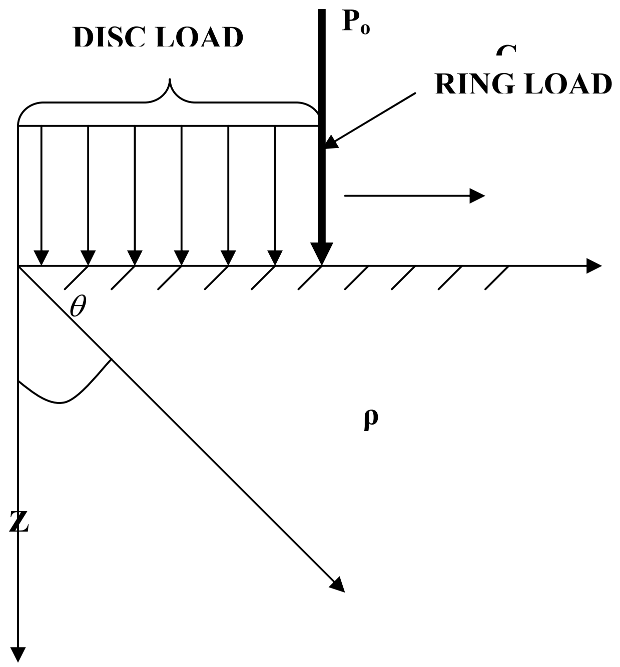

Fig. a. Expending ring and disk loads

We consider surface loads (ring or disc depicted in Fig. a), acting normal to the surface, emanates from the origin of the coordinates and expands radially at constant rate c over the surface.
Since we are considering two dimensional problem, so we assume the components of displacement vector $\boldsymbol{u_i}$ of the form

$$\boldsymbol{u_i} = (u_r^{\,i}, 0, u_z^{\,i}) \quad \text{where} \quad i=F,S \tag{6}$$

in equations (1)-(5) where $(u_r^{\,i}, u_z^{\,i})$ are radial and vertical components of $\boldsymbol{u_i}$.

For further consideration it is convenient to introduce the dimensionless quantities defined as:

$$(z', r') = \frac{\omega^*}{C_1}(z, r), t' = \omega^* t, \ (p', F_0') = (\frac{p}{E}, \frac{F_0}{E}) \ (u_r'^S, u_r'^F) = (u_r^{\,S}, u_r^{\,F}) \left[\frac{\lambda^S + 2\mu^S}{E}\right]\frac{\omega^*}{C_1},$$

$$(u_z'^S, u_z'^F) = (u_z^{\,S}, u_z^{\,F}) \left[\frac{\lambda^S + 2\mu^S}{E}\right]\frac{\omega^*}{C_1}, (t_{zz}', t_{zr}') = (t_{zz}, t_{zr})\frac{1}{E} \ . \tag{7}$$

In these relations E is the Young's modulus of the solid phase, ω^* is a constant having the dimensions of frequency, C_1 is the velocity of a longitudinal wave propagating in a fluid saturated incompressible porous medium and is given by

$$C_1 = \sqrt{\frac{(\eta^F)^2(\lambda^S + 2\mu^S)}{(\eta^F)^2\rho^S + (\eta^S)^2\rho^F}}, \tag{8}$$

If pore is absent or gas is filled in the pores then ρ^F is very small as compare to ρ^S and can be neglected so the relation reduce to

$$C_0 = \sqrt{\frac{\lambda^S + 2\mu^S}{\rho^S}} \ . \tag{9}$$

We assume the time harmonic behaviour as

$$(u_r^{\,i}, u_z^{\,i}, p)(r,z,t) = (u_r^{\,i}, u_z^{\,i}, p)((r,z)\ e^{i\omega t}, \ i = F,S \tag{10}$$

The displacement components $u_r^{\,i}$ and $u_z^{\,i}$ (i = F,S) are related to the non dimensional potentials ϕ^i and ψ^i as

$$u_r^{\,i} = \frac{\partial\phi^i}{\partial r} - \frac{\partial\psi^i}{\partial z}, \ u_z^{\,i} = \frac{\partial\phi^i}{\partial z} + \frac{\partial\psi^i}{\partial r} + \frac{\psi^i}{r} \ . \tag{11}$$

We define the Hankel transform as follows:

$$\hat{f}\,(\xi, z) = \int_0^\infty f(r,z)\ r\ J_n(r\xi) dr\ , \tag{12}$$

where J_n (x) is the Bessel function of the first kind of index n.

With the aids of (1)–(3) and (10)-(12), the values of $\hat{\phi}^S$, $\hat{\phi}^F$, $\hat{p}$, $\hat{\psi}^S$, $\hat{\psi}^F$ satisfying radiation conditions that $\hat{\phi}^S$, $\hat{\phi}^F$, $\hat{p}$, $\hat{\psi}^S$, $\hat{\psi}^F \to 0$ as z $\to \infty$ are given by

$$(\hat{\phi}^S, \hat{\phi}^F, \hat{p}) = (1, -\frac{\eta^S}{\eta^F}, m_1) A_1 e^{-a_1 z}, (\hat{\psi}^S, \hat{\psi}^F) = (1, n_1) B_1 e^{-a_2 z} \ , \tag{13}$$

where

$$a_1^{\,2} = \xi^2 - \omega^2 + \frac{\delta^2}{(\eta^F)^2}, a_2^{\,2} = \frac{\xi^2\,\delta^2 - \delta_1^{\,2}\omega^2 - \delta_2 i\omega + \delta_2 i\omega\, n_1}{\delta^2},\ \ n_1 = \frac{i\omega\,\delta_2}{-\dfrac{\rho^F}{\rho^S}\delta_1^{\,2}\,\omega^2 + i\omega\delta_2},$$

$$m_1 = \frac{1}{(\eta^F)^2}\left(-\frac{\rho^F}{\rho^S}\eta^S\delta_1^{\,2}\omega^2 + i\omega\delta_2\right),\ \delta_1 = \frac{C_1}{C_O},\ \delta = \frac{\beta_0}{C_O},\ \beta_O = \sqrt{\frac{\mu^S}{\rho^S}},\ \delta_2 = \frac{S_V C_1^{\,2}}{w^*\rho^S C_O^{\,2}}.$$

4. Boundary conditions

The boundary conditions at the surface z = 0 are

$t_{zz}^{\ S}$ -p = - $P_0(r,t)$, $t_{zr}^{\ S}$ = 0, at z = 0 (14)

where

$$P_0(r,t) = \begin{cases} \dfrac{F_0}{2\pi r}\delta(ct-r)e^{i\omega t}, & \textit{for ring load} \\ \dfrac{F_0}{\pi(ct)^2}H(ct-r)e^{i\omega t}, & \textit{for disc load} \end{cases} \tag{15}$$

in which δ( ) is the Dirac delta function and H( ) is the heaviside function and F_0 is the magnitude of the force.

Using Hankel transforms defined by (12), we obtain,

$$\hat{P}_0(\xi) = \begin{cases} \dfrac{F_0}{2\pi}\dfrac{1}{\sqrt{\xi^2 - \dfrac{\omega^2}{c^2}}} & \textit{for ring load} \\ \dfrac{F_0}{\pi c\,\xi}\left(\sqrt{(\xi^2 - \dfrac{\omega^2}{c^2})} - \dfrac{i\omega}{c}\right) & \textit{for disc load} \end{cases} \tag{16}$$

With the aid of (10) - (15), we obtain the components of displacement, stress and pore pressure as :

$$(\hat{u}_z^{\ S}, \hat{u}_z^{\ F}) = \frac{1}{\Delta}\sum_{i=1}^{2}(\Delta_{3i}, \Delta_{4i})e^{-a_i z},\ (\hat{t}_{zz}, \hat{t}_{zr}) = \frac{1}{\Delta}\sum_{i=1}^{2}(\Delta_{5i}, \Delta_{6i})e^{-a_i z},\ \hat{p} = \frac{1}{\Delta}(\ \Delta_{71}\ e^{-a_1 z}), \tag{17}$$

where

$$\Delta = l_2 l_3 - l_1 l_4,\ (\Delta_{31}, \Delta_{32}) = (a_1\, l_4, \xi\, l_3)\hat{P}_0(\xi),\ (\Delta_{41}, \Delta_{42}) = (-\frac{\eta^S}{\eta^F}\, a_1\, l_4, \xi\ n_1\, l_3)\hat{P}_0(\xi),$$

$(\Delta_{51},\Delta_{52})=(-l_1\, l_4 - m_1\, l_4, -l_2\, l_3)\hat{P}_0(\xi), (\Delta_{61},\Delta_{62}) = (-1,-1)\, l_3\, l_4\, \hat{P}_0(\xi), \Delta_{71} = -m_1\, l_4\, \hat{P}_0(\xi),$

$l_1 = (2\delta^2\xi^2 + a_1^{\,2} - \xi^2) - m_1, (l_2, l_3) = (a_2, a_1)\, 2\xi\delta^2, \; l_4 = \delta^2(a_2^{\,2} + \xi^2).$

The expressions for displacement, stress and pore pressure for ring load and disc load are obtained by substituting the values of $\hat{P}_0(\xi)$ from equation (16) in equation (17).

5. Particular case

If the pore liquid is absent, the corresponding expressions in empty porous medium are

$$(\hat{u}_z^{\,S}, \hat{t}_{zz}, \hat{t}_{zr}) = \frac{1}{\Delta^\bullet}\sum_{i=1}^{2}(\Delta^\bullet_{3i}, \Delta^\bullet_{5i}, \Delta^\bullet_{6i})e^{-a_i^\bullet z}), \tag{18}$$

where

$a_1^{\bullet 2} = \xi^2 - \omega^2, a_2^{\bullet 2} = \xi^2 - \dfrac{\omega^2}{\delta^2}, \Delta^\bullet = l_2^\bullet\, l_3^\bullet - l_1^\bullet\, l_4^\bullet, (\Delta^\bullet_{31}, \Delta^\bullet_{32}) = (a_1^\bullet\, l_4^\bullet, \xi\, l_3^\bullet)\, \hat{P}_0(\xi)$

$(\Delta^\bullet_{51}, \Delta^\bullet_{52}, \Delta^\bullet_{61}, \Delta^\bullet_{62}) = (-l_1^\bullet\, l_4^\bullet, -l_2^\bullet\, l_3^\bullet, -l_3^\bullet\, l_4^\bullet, -l_3^\bullet\, l_4^\bullet)\hat{P}_0(\xi)\; l_1^\bullet = (2\delta^2\xi^2 + a_1^{\bullet 2} - \xi^2),$

$l_2^\bullet = 2a_2^\bullet\xi\delta^2, \; l_3^\bullet = 2a_1^\bullet\xi\delta^2, \; l_4^\bullet = \delta^2(a_2^{\bullet 2} + \xi^2).$

6. Inversion of the transform

To obtain the solution of the problem in the physical domain, we invert the Hankel transform by using the method described by Kumar et. al. [8].

7. Numerical results and discussion

Following Boer and Ehlers [5] we take the values of the various physical parameters as:

$\eta^S = .67$, $\eta^F = .33$, $\rho^S = 1.34\text{Mg}/m^3$, $\rho^F = .33\text{Mg}/m^3$, $k^F = .01\text{m/s}$,

$\lambda^S = 5.5833\text{MN}/m^2$, $\gamma^{FR} = 10.00\text{KN}/m^3$, $\mu^S = 8.3750\ \text{MN}/m^2$.

The values of normal solid stress t_{zz}, tangential solid stress t_{zr} and pore pressure p for fluid saturated incompressible porous medium (FM) and empty porous medium (EM) are shown due to ring and disc loads in figs. 1-6. The computations are carried out for two values of dimensionless frequency ω = 0.05 and ω = 0.1 at z =1 in the range $0 \le r \le 10$ and c = 1.

Also the solid and small dashed line without central symbol correspond to the variations at ω = 0.05 and solid and small dashed line with central symbol (- o - o -) correspond to the variations at ω = 0.1. Curves without and with central symbols correspond to the case of FM and EM respectively.

Figs.1 and 3 shows the variations of normal solid stress t_{zz} with distance r due to ring and disc loads. Values of t_{zz} for FM start with small initial decrease then oscillates near the application of load for all the values of r, whereas for EM its values initially increase sharply untimely become close to zero in an oscillatory manner for both the frequencies. Near the application of load values of t_{zz} for FM are greater than that for EM and its values for FM in Figs.1 and 3 are magnified by 10^2.
Behaviour of tangential solid stress t_{zr} due to ring and disc loads is shown in the figs. 2 and 4.Values of t_{zr} for FM oscillates about origin for the whole range of r, whereas for EM its values decrease monotonically in the range $0 \le r \le 2.7$, then oscillates about origin as r increase further for both the frequencies. Near the application of load values of t_{zr} for FM increase with the increase in the value of frequency whereas reverse behaviour is observed for EM and its values for FM in Figs.2 and 4 are magnified by 10^2.
Figs.5 and 6 depicts the variations of the pore pressure p with distance r due to ring and disc loads. Values of p increase sharply in the range $0 \le r \le 2.5$, decrease in the range $2.5 \le r \le 4$, increase in the range $4 \le r \le 5.3$, untimely become close to zero afterwards for both the frequencies.

8. Conclusion

Appreciable porosity effect is observed on normal and tangential stresses on the application of ring and disc loads for different frequencies. The trends of variations of all the quantities are similar for both types of loads with difference in magnitude of values of the corresponding quantity.

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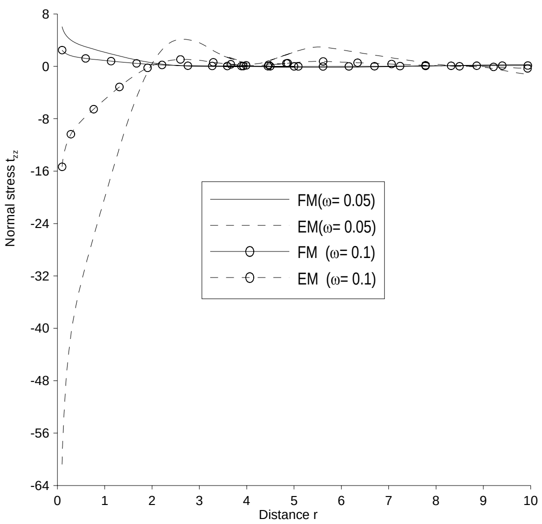

Fig.1 Variation of normal stress t_{zz} with distance r due to ring load

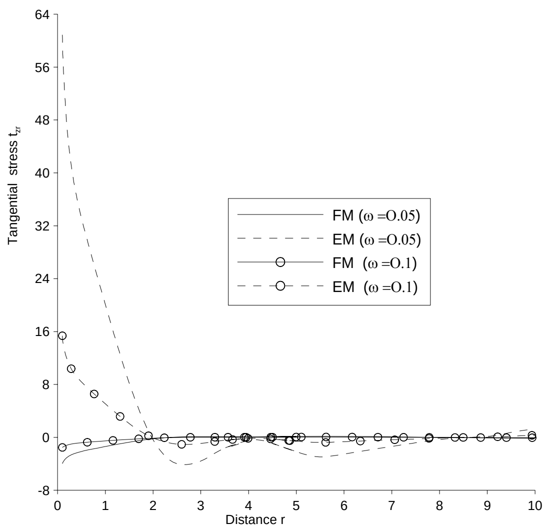

Fig.2 Variation of tangential stress t_{zr} with distance r due to ring load

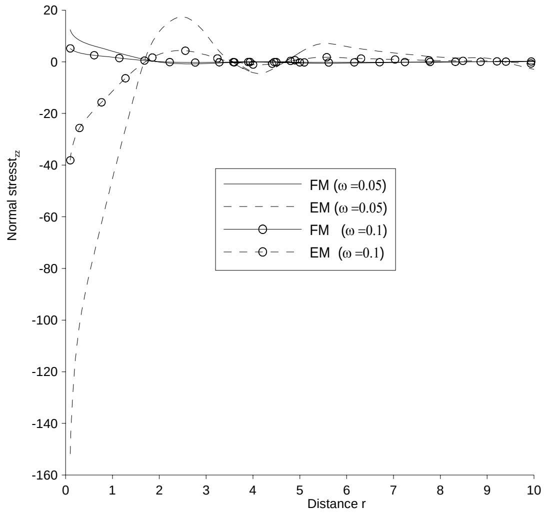

Fig.3 Variation of normal stresst$_{zz}$ with distance r due to disc load

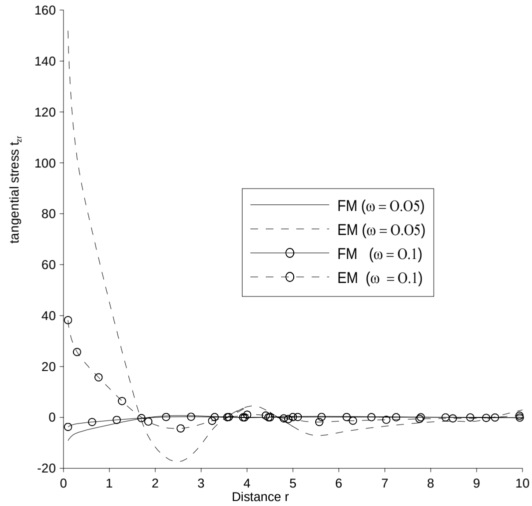

Fig.4 Variation of tangential stress t$_{zr}$ with distance r due to disc load

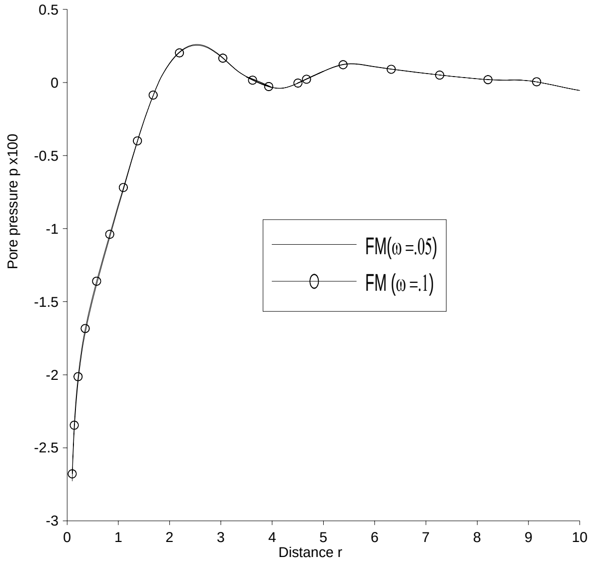

Fig. 5 Variation of pore pressure p with distance r due to ring load

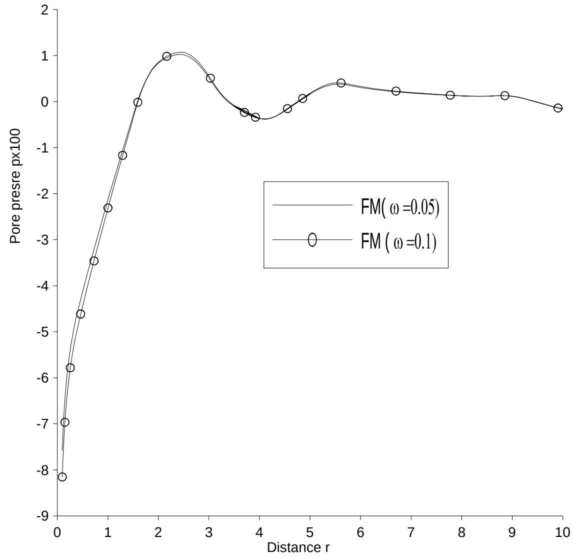

Fig.6 Variation of pore presre p with distance r due to disc load

Received: November, 2009