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Convergence of Iterative Methods for Solving Painlevé Equation

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Abstract

In this paper, a Painlevé equation is solved by using the Adomian's decomposition method (ADM) , modified Adomian's decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM), homotopy perturbation method (HPM), modified homotopy perturbation method (MHPM) and homotopy analysis method (HAM). The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords: Painlevé equation, Adomian decomposition method , Modified Adomian decomposition method , Variational iteration method , Modified variational iteration method ,Homotopy perturbation method , Modified homotopy perturbation method ,Homotopy analysis method

1 Introduction

The Painlevé equation and their solutions arise in parts of pure and applied mathematics and theoretical physics such as the second Painlevé equation as a model for the electric field in a semiconductor [17]. Painlevé considered wide class of second order equations and classified them to the nature of singularities. Painlevé and his coworkers found essentially six different equations within the class considered whose solutions are single valued as functions of complex independent variables, except possibly at the fixed singularities of the coefficients. These are known as Painlevé transcendents and have a great variety of interesting properties and applications. A lot of works have been done in order to find the numerical solution of this equation. For example, reduction of KdV and cylindrical KdV equations to Painlevé equation [12], numerical studies of the fourth Painlevé equation [1], variational iteration method and homotopy perturbation method [5], the numerical solution of the second Painlevé equation [13], on comparison for series and numerical solutions for second Painlevé equation [18]. In this work, we develop the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve the Painlevé equation as follows:

$$\frac{d^2u}{dx^2} = 6u^2(x) + x, \tag{1}$$

with the initial conditions given by:

$$\begin{aligned} &u(0) = 0, \\ &u^{'}(0) = 1. \end{aligned}$$

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved in section 3. Finally, the numerical example is presented in section 4 to illustrate the accuracy of these methods.

To obtain the approximate solution of Eq.(1), by integrating 2 times from Eq.(1) with respect to x and using the initial conditions we obtain,

$$u(x) = G(x) + 6\int_0^x \int_a^x F(u(x))dxdx, \tag{2}$$

where,

$$\begin{aligned} &G(x) = x + \tfrac{1}{6}x^3, \\ &F(u(x)) = 6u^2(x). \end{aligned}$$

The double integrals in (2) can be written as [2]:

$$\int_0^x \int_0^x F(u(x))dxdx = \int_0^x (x-t)F(u(t))dt.$$

So, we can write Eq.(2) as follows:

$$u(x) = G(x) + 6\int_0^x (x-t)F(u(t))dt. \tag{3}$$

Now we decompose the unknown function $u(x)$ by the following decomposition series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x).$$

In Eq.(3), we assume $G(x)$ is bounded for all x in $J = [0, T]$ and

$$\mid x - t \mid \leq M^{'}, \quad \forall\ 0 \leq x, t \leq T.$$

We assume the term $F(u)$ is Lipschitz continuous with $\mid F(u) - F(u^*) \mid \leq L \mid u - u^* \mid$ and

$$\alpha = TM^{'}L,$$

$$\gamma = 1 - T^2\alpha,$$

$$\beta = 1 - T^2(M^{'}(1 - \alpha)).$$

2 The iterative methods

2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g(x), \tag{4}$$

where u is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(4), and using the given conditions we obtain

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{5}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$. The nonlinear operator $Nu = H_1(u)$ is decomposed as

$$H_1(u) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}. \tag{7}$$

Adomian polynomials were introduced in [19,14,3] as

$$\begin{array}{ll} A_0 & = H_1(u_0), \\ A_1 & = u_1 H_1'(u_0), \\ A_2 & = u_2 H_1'(u_0) + \frac{1}{2!}u_1^2 H_1''(u_0), \\ A_3 & = u_3 H_1'(u_0) + u_1 u_2 H_1''(u_0) + \frac{1}{3!}u_1^3 H_1'''(u_0), ... \end{array} \tag{8}$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of u in (4) as the following series,

$$u = \sum_{i=0}^{\infty} u_i, \tag{9}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$\begin{array}{l} u_0 = G(x), \\ u_1 = \int_0^x (x-t)\ A_0(t)\ dt, \\ \vdots \\ u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n) = \int_0^x (x-t)\ A_n(t) dt, \quad n \geq 0. \end{array} \tag{10}$$

Substituting (8) into (10) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by (9) follows immediately.

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [2]. The modified forms was established based on the assumption that the function $G(x)$ can be divided into two parts, namely $G_1(x)$ and $G_2(x)$. Under this assumption we set

$$G(x) = G_1(x) + G_2(x). \tag{11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part G_1 be assigned to the zeroth component u_0, whereas the remaining part G_2 be combined with the other terms given in (10) to define u_1. Consequently, the modified recursive relation

$$\begin{array}{ll} u_0 & = G_1(x), \\ u_1 & = G_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ & \vdots \\ u_{n+1} & = -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{array} \tag{12}$$

was developed.

To obtain the approximation solution of Eq.(1), according to the MADM, we can write the iterative formula (12) as follows:

$$\begin{array}{l} u_0(x) = G_1(x), \\ u_1(x) = G_2(x) + \int_0^x (x-t)\ A_0(t)dt, \\ \vdots \\ u_{n+1}(x) = \int_0^x (x-t)\ A_n(t)dt. \end{array} \tag{13}$$

The operator $F(u(x))$ is usually represented by the infinite series of the Adomian polynomials as follows:

$$F(u) = \sum_{i=0}^{\infty} A_i,$$

where $A_i, i \geq 0$ are the Adomian polynomials.

Also, we can use the following formula for the Adomian polynomials [6]:

$$A_n = F(s_n) - \sum_{i=0}^{n-1} A_i. \tag{14}$$

Where the partial sum is $s_n = \sum_{i=0}^{n} u_i(x)$.

2.2 Description of the VIM and MVIM

In the VIM [8-11], we consider the following nonlinear differential equation:

$$L(u) + N(u) = g(t), \tag{15}$$

where L is a linear operator,N is a nonlinear operator and $g(t)$ is a known analytical function. In this case, a correction functional can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\{L(u_n(\tau)) + N(u_n(\tau)) - g(\tau)\}d\tau, \quad n \geq 0, \quad (16)$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $u_n(\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n\to\infty} u_n(t). \quad (17)$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq.(1), according to the VIM, we can write iteration formula (16) as follows:

$$u_{n+1}(x) = u_n(x) + L_t^{-1}(\lambda(x)[u_n(x) - G(x) - \int_0^x (x-t)\ F(u_n(t))\ dt]), \quad (18)$$

where,

$$L_t^{-1}(.) = \int_0^x \int_0^x (.)\ dt\ dt.$$

To find the optimal $\lambda(x)$, we proceed as

$$\delta u_{n+1}(x) = \delta u_n(x,t) + \delta L_t^{-1}(\lambda(x)[u_n(x) - G(x) - \textstyle\int_0^x (x-t)\ F(u_n(t))\ dt]). \quad (19)$$

From Eq.(19), the stationary conditions can be obtained as follows:

$1 - \lambda^{'}(x)\ |_{x=t} = 0$ and $\lambda^{''}(x)\ |_{x=t} = 0$.

Therefore, the Lagrange multipliers can be identified as $\lambda(x) = x - t$ and by substituting in (18), the following iteration formula is obtained.

$$u_0(x) = G(x),$$
$$u_{n+1}(x) = u_n(x,t) + L_t^{-1}((x-t)[u_n(x,t) - G(x) - \int_0^x (x-t)\ F(u_n(t))\ dt, n \geq 0. \tag{20}$$

To obtain the approximation solution of Eq.(1), based on the MVIM [25,26,20], we can write the following iteration formula:

$$u_0(x) = G(x),$$
$$u_{n+1}(x) = u_n(x) + L_t^{-1}((x-t)[\int_0^x (x-t)\ F(u_n(t) - u_{n-1}(t))\ dt]), n \geq 0. \tag{21}$$

Relations (20) and (21) will enable us to determine the components $u_n(x)$ recursively for $n \geq 0$.

2.3 Description of the HAM

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x)$ is unknown function and x is an independent variable. let $u_0(x)$ denote an initial guess of the exact solution $u(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary nonlinear operator with the property $L[r(x)] = 0$ when $r(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x;q) - u_0(x)] - qhH(x)N[\phi(x;q)] = \hat{H}[\phi(x;q); u_0(x), H(x), h, q]. \tag{22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x)$, the auxiliary nonlinear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x)$.

Enforcing the homotopy (22) to be zero, i.e.,

$$\hat{H}[\phi(x;q); u_0(x), H(x), h, q] = 0, \tag{23}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q)-u_0(x)] = qhH(x)N[\phi(x;q)]. \quad (24)$$

When $q = 0$, the zero-order deformation Eq.(24) becomes

$$\phi(x;0) = u_0(x), \quad (25)$$

and when $q = 1$, since $h \neq 0$ and $H(x) \neq 0$, the zero-order deformation Eq.(24) is equivalent to

$$\phi(x;1) = u(x). \quad (26)$$

Thus, according to (25) and (26), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $u_0(x)$ to the exact solution $u(x)$. Such a kind of continuous variation is called deformation in homotopy [21,22].

Due to Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m, \quad (27)$$

where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;q)}{\partial q^m} \mid_{q=0} .$$

Let the initial guess $u_0(x)$, the auxiliary nonlinear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x)$ be properly chosen so that the power series (27) of $\phi(x;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x) = \phi(x;1) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \quad (28)$$

From Eq.(27), we can write Eq.(24) as follows

$$(1-q)L[\phi(x,q)-u_0(x)] = (1-q)L[\textstyle\sum_{m=1}^{\infty} u_m(x)\ q^m] = q\ h\ H(x)N[\phi(x,q)] \Rightarrow$$
$$L[\textstyle\sum_{m=1}^{\infty} u_m(x)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x)q^m] = q\ h\ H(x)N[\phi(x,q)] \quad (29)$$

By differentiating (29) m times with respect to q, we obtain

$\{L[\sum_{m=1}^{\infty} u_m(x)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x)q^m]\}^{(m)} = \{q\ h\ H(x)N[\phi(x,q)]\}^{(m)} = m!\ L[u_m(x) - u_{m-1}(x)] = h\ H(x)\ m\ \frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}\ |_{q=0}$.

Therefore,

$$\begin{aligned} &L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)\Re_m(u_{m-1}(x)),\\ &u_m(0) = 0, \end{aligned} \tag{30}$$

where,

$$\Re_m(u_{m-1}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{31}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(30) is governing the nonlinear operator L, and the term $\Re_m(u_{m-1}(x))$ can be expressed simply by (31) for any nonlinear operator N.

To obtain the approximation solution of Eq.(1), according to HAM, let

$$N[u] = u(x) - G(x) - \int_0^x (x-t)F(u(t))dt,$$

so

$$\Re_m(u_{m-1}(x)) = u_{m-1}(x) - \int_0^x (x-t)F(u_{m-1}(t))dt - (1-\chi_m)G(x)\ . \tag{32}$$

Substituting (32) into (30)

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)[u_{m-1}(x) - \int_0^x (x-t)F(u(t))dt - (1-\chi_m)G(x)]. \tag{33}$$

We take an initial guess $u_0(x) = G(x)$, an auxiliary nonlinear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x) = 1$. This is substituted into (33) to give the recurrence relation

$$\begin{aligned} &u_0(x) = G(x),\\ &u_n(x) = \int_0^x (x-t)F(u_{n-1}(t))\ dt, \quad n \geq 1. \end{aligned} \tag{34}$$

Therefore, the solution $u(x)$ becomes

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = G(x) + \sum_{n=1}^{\infty} \left(\int_0^x (x-t) F(u_{n-1}(t)) \, dt \right). \tag{35}$$

Which is the method of successive approximations. If

$$\mid u_n(x) \mid < 1,$$

then the series solution (35) convergence uniformly.

2.4 Description of the HPM and MHPM

[23,15] To explain HPM we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u), \tag{36}$$

according to HPM, we construct a homotopy which satisfies the following relation

$$H(u,p) = Lu - Lv_0 + pLv_0 + p[Nu - f(u)] = 0, \tag{37}$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq.(37) is expressed as

$$u(x) = u_0(x) + pu_1(x) + p^2 u_2(x) + ... \tag{38}$$

hence the approximation solution of Eq.(36) can be expressed as a series of the power of p, i.e.

$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + ... \tag{39}$$

$$\begin{aligned} u_0(x) &= G(x), \\ u_m(x) &= \sum_{k=0}^{m-1} \int_0^x (x-t) F(u_{m-k-1}(t)) dt \quad m \geq 1. \end{aligned} \tag{40}$$

To explain MHPM, we consider Eq. (1) as

$$L(u) = u(x) - G(x) - \int_0^x (x-t) F(u_{n-1}(t)) dt.$$

We can define homotopy $H(u,p,m)$ by

$$H(u,o,m) = f(u), \quad H(u,1,m) = L(u).$$

Where m is an unknown real number and

$$f(u) = u(x) - G(x).$$

Typically we may choose a convex homotopy by

$$H(u,p,m) = (1-p)f(u) + pL(u) + p(1-p)[m(F(u))] = 0, \qquad 0 \le p \le 1. \tag{41}$$

where m is called the accelerating parameters, and for $m = 0$ we define $H(u,p,0) = H(u,p)$, which is the standard HPM.

The convex homotopy (41) continuously trace an implicity defined curve from a starting point $H(u(x)-f(u),0,m)$ to a solution function $H(u(x),1,m)$. The embedding parameter p monotonically increase from o to 1 as trivial problem $f(u) = 0$ is continuously deformed to original problem $L(u) = 0$. [24,16,4]

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \tag{42}$$

when $p \to 1$ Eq. (37) corresponds to the original one, Eq. (42) becomes the approximate solution of Eq. (1), i.e.,

$$u = \lim_{p\to 1} v = \textstyle\sum_{n=0}^{\infty} u_n, \tag{43}$$

where,

$$\begin{array}{l} u_0(x,t) = G(x), \\ u_1(x,t) = \sum_{k=0}^{m-1} \int_0^x (x-t)F(u_{n-1}(t))dt - m(F(u(x))(x)), \\ u_2(x,t) = \sum_{k=0}^{m-1} \int_0^x (x-t)F(u_{n-1}(t))dt + m(F(u(x))(x)), \\ \vdots \\ u_m(x,t) = \sum_{k=0}^{m-1} \int_0^x (x-t)F(u_{m-k-1}(t))dt,\ m \ge 3. \end{array} \tag{44}$$

3 Existence and convergency of iterative methods

Theorem 3.1. *Let $0 < \alpha < 1$, then nonlinear Painlevé equation (1), has a unique solution.*

Proof. *Let u and u^* be two different solutions of (3) then*

$$\mid u-u^* \mid = \mid \int_0^x (x-t)\ [F(u(t))-F(u^*(t))]\ dt \mid$$
$$\leq \int_0^x \mid (x-t) \mid\mid F(u(t))-F(u^*(t)) \mid\ \ dt$$
$$\leq TM^{'}L\ \ \mid u-u^* \mid = \alpha \mid u-u^* \mid$$

From which we get $(1-\alpha) \mid u-u^* \mid \leq 0$. *Since* $0<\alpha<1$. *then* $\mid u-u^* \mid = 0$. *Implies* $u=u^*$ *and completes the proof.* □

Theorem 3.2. *The series solution* $u(x)=\sum_{i=0}^{\infty} u_i(x)$ *of problem(1) using MADM convergence when*

$0<\alpha<1$, $\mid u_1(x) \mid < \infty$.

Proof. *Denote as* $(C[J], \parallel . \parallel)$ *the Banach space of all continuous functions on* J *with the norm* $\parallel f(t) \parallel = max \mid f(t) \mid$, *for all* t *in* J. *Define the sequence of partial sums* s_n, *let* s_n *and* s_m *be arbitrary partial sums with* $n \geq m$. *We are going to prove that* s_n *is a Cauchy sequence in this Banach space:*

$$\parallel s_n - s_m \parallel = max_{\forall t\in J} \mid s_n - s_m \mid = max_{\forall t\in J} \mid \sum_{i=m+1}^{n} u_i(x) \mid$$
$$= max_{\forall t\in J} \mid \sum_{i=m+1}^{n} \int_0^x (x-t)A_{i-1}\ dt \mid = max_{\forall t\in J} \mid \int_0^x\ (x-t)(\sum_{i=m}^{n-1} A_i)\ dt \mid .$$

From [6], we have

$$\sum_{i=m}^{n-1} A_i = F(s_{n-1}) - F(s_{m-1}).$$

So,

$$\parallel s_n - s_m \parallel = max_{\forall t\in J} \mid \int_0^x (x-t)[F(s_{n-1})-F(s_{m-1})]\ dt \mid \leq$$
$$\int_0^x \mid (x-t) \mid\mid F(s_{n-1})-F(s_{m-1}) \mid\ \ dt \leq \alpha \parallel s_n - s_m \parallel .$$

Let $n=m+1$, *then*

$$\parallel s_n - s_m \parallel \leq \alpha \parallel s_m - s_{m-1} \parallel \leq \alpha^2 \parallel s_{m-1} - s_{m-2} \parallel \leq ... \leq \alpha^m \parallel s_1 - s_0 \parallel .$$

From the triangle inquality we have

$$\parallel s_n - s_m \parallel \leq \parallel s_{m+1} - s_m \parallel + \parallel s_{m+2} - s_{m+1} \parallel + ...$$
$$+ \parallel s_n - s_{n-1} \parallel \leq [\alpha^m + \alpha^{m1} + ... + \alpha^{n-m-1}] \parallel s_1 - s_0 \parallel$$
$$\leq \alpha^m[1+\alpha+\alpha^2+...+\alpha^{n-m-1}] \parallel s_1 - s_0 \parallel \leq \alpha^m[\tfrac{1-\alpha^{n-m}}{1-\alpha}] \parallel u_1(x) \parallel .$$

Since $0<\alpha<1$, *we have* $(1-\alpha^{n-m})<1$, *then*

$$\| s_n - s_m \| \leq \frac{\alpha^m}{1-\alpha} max_{\forall t \in J} \mid u_1(x) \mid .$$

But $\mid u_1(x) \mid < \infty$ *, so, as* $m \to \infty$*, then* $\| s_n - s_m \| \to 0$*. We conclude that* s_n *is a Cauchy sequence in* $C[J]$*, therefore the series is convergence and the proof is complete.* □

Theorem 3.3. *The series solution* $u(x) = \sum_{i=0}^{\infty} u_i(x)$ *of problem (1) using VIM converges when* $0 < \alpha < 1$ *,* $0 < \beta < 1$*.*

Proof.

$$u_{n+1}(x) = u_n(x) - L_t^{-1}((t-x)[u_n(x) - G(x) - \int_0^x (x-t)F(u_n(t))\ dt]) \tag{45}$$

$$u(x) = u(x) - L_t^{-1}((t-x)[u(x) - G(x) - \int_0^x (x-t)F(u(t))\ dt]) \tag{46}$$

By subtracting relation (45) from (46),

$$\begin{aligned} &u_{n+1}(x) - u(x) = u_n(x) - u(x) \\ &-L_t^{-1}((t-x)(u_n(x) - u(x) - \int_0^x (x-t)[F(u_n(t)) - F(u(t))]\ dt)), \end{aligned}$$

if we set, $e_{n+1}(x) = u_{n+1}(x) - u_n(x)$*,* $e_n(x) = u_n(x) - u(x)$*, then*

$$\begin{aligned} &e_{n+1}(x) = e_n(x) - L_t^{-1}((t-x)(e_n - \int_0^x (x-t)[F(u_n(t)) - F(u(t))]\ dt) \\ &\leq e_n - T^2(M'(\mid e_n \mid - TM'L\ e_n)) \end{aligned}$$

If $e_n > 0$ *then* $\mid e_n \mid = e_n$ *so we have*

$$e_{n+1} \leq e_n\ (1 - T^2(M'(1-\alpha))) = e_n \beta$$

Therefore,

$$\|e_{n+1}\| = max_{\forall t \in J} \mid e_{n+1} \mid \leq \beta\ max_{\forall t \in J} \mid e_n \mid \leq \beta \|e_n\|.$$

Since $0 < \beta < 1$*, then* $\|e_n\| \to 0$*. So, the series converges and the proof is complete.* □

Theorem 3.4. *If the series solution (34) of problem (1) using HAM convergent then it converges to the exact solution of the problem (1).*

Proof. *We assume:*

$$u(x) = \sum_{m=0}^{\infty} u_m(x),$$
$$\widehat{F}(u(x)) = \sum_{m=0}^{\infty} F(u_m(x)).$$

where

$$\lim_{m\to\infty} u_m(x) = 0.$$

We can write,

$$\sum_{m=1}^{n}[u_m(x) - \chi_m u_{m-1}(x)] = u_1 + (u_2 - u_1) + ... + (u_n - u_{n-1}) = u_n(x). \quad (47)$$

Hence, from (47),

$$\lim_{n\to\infty} u_n(x) = 0. \quad (48)$$

So, using (48) and the definition of the nonlinear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = L[\sum_{m=1}^{\infty}[u_m(x) - \chi_m u_{m-1}(x)]] = 0.$$

therefore from (30), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x)) = 0.$$

Since $h \neq 0$ and $H(x) \neq 0$, we have

$$\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x)) = 0. \quad (49)$$

By substituting $\Re_{m-1}(u_{m-1}(x))$ into the relation (49) and simplifying it , we have

$$\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x)) = \sum_{m=1}^{\infty}[u_{m-1}(x) - \int_0^x (x-t)F(u_{m-1}(t))dt$$
$$-(1-\chi_m)G(x)] = u(x) - G(x) - \int_0^x (x-t)\widehat{F}(u_{m-1}(t))dt. \quad (50)$$

From (49) and (50), we have

$$u(x) = G(x) + \int_0^x (x-t)\widehat{F}(u_{m-1}(t))dt,$$

therefore, $u(x)$ must be the exact solution. □

Theorem 3.5. *The series solution $u(x) = \sum_{i=0}^{\infty} u_i(x)$ of problem (3) using HPM converges [7].*

Theorem 3.6. *The series solution (21) of problem (1) using MVIM converges when $0 < \alpha < 1$, $0 < \gamma < 1$.*

Remark 1. Proving of convergence the MHPM is similar to proving of convergence HPM.

Remark 2. Proving of convergence the MVIM is similar to proving of convergence VIM.

4 Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM,HPM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm. In this algorithm ε is a given positive value.

Algorithm:

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relation (10) for ADM, (13) for MADM, (20) for VIM, (21) for MVIM, (34) for HAM, (40) for HPM and (44) for MHPM.

Step 3. If $| u_{n+1} - u_n | < \varepsilon$ then go to step 4,
else $n \leftarrow n + 1$ and go to step 2.

Step 4. Print $u(x) = \sum_{i=0}^{n} u_i(x)$ as the approximate of the exact solution.

Example 4.1. Consider the Painlevé equation [5],

Table 1. Numerical results for Example 1

x				
	ADM(n=9)	MADM(n=6)	VIM(n=4)	MVIM(n=3)
0.1	0.1002158432	0.1002601271	0.1002167477	0.1002167477
0.2	0.2021177613	0.2021288956	0.2021394527	0.2021394527
0.3	0.3086306243	0.3086306987	0.3086307490	0.3086307490
0.4	0.4239851458	0.4239860367	0.4239862788	0.4239862874
0.5	0.5543356243	0.5543370146	0.5543399110	0.5543400236
0.7	0.8992174326	0.8992199875	0.8992296944	0.8992316654
0.9	1.4812013434	1.4814889672	1.4817789515	1.4819978846
1.0	1.9371446837	1.9416721356	1.9594210425	1.9616038852

x			
	HPM(n=4)	MHPM(n=2)	HAM(n=4)
0.1	0.1002167477	0.1002167477	0.1002164834
0.2	0.2021394527	0.2021394527	0.2021354316
0.3	0.3086307492	0.3086307492	0.3086307127
0.4	0.4239862896	0.4239862912	0.4239862745
0.5	0.5543401181	0.5543401896	0.5543384471
0.7	0.8992493820	0.8992199875	0.8992237234
0.9	1.4824443771	1.4826178496	1.4822596307
1.0	1.9624521863	1.9718875613	1.9621478174

Table 1 shows that, approximate solution of the nonlinear Painlevé equation is convergence with 2 iterations by using the MHPM. By comparing the results of table 1 , we can observe that the MHPM is more rapid convergence than the ADM, MADM, VIM, MVIM and HAM.

5 Conclusion

The MHPM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the MHPM has been successfully employed to obtain the approximate analytical solution of the Painlevé equation. For this purpose, we showed that the MHPM is more rapid convergence than the ADM, MADM, VIM,MVIM and HAM.

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