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Applications of Fractional Calculus

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Abstract

Different definitions of fractional derivatives and fractional Integrals (Differintegrals) are considered. By means of them explicit formula and graphs of some special functions are derived. Also we reviw some applications of the theory of fractional calculus.

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1 Introduction

Fractional calculus is a field of mathematics study that qrows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value.

The concept of fractional calculus(fractional derivatives and fractional integral) is not new. In 1695 *L'Hospital* asked the question as to the meaning of $d^n y/dx^n$ if $n = 1/2$; that is " what if n is fractional?". *Leibniz* replied that "$d^{1/2}x$ will be equal to $x\sqrt{dx : x}$".

It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, which represent a rapidly qrowing field both in theory and in applications to real world problems, it is not so. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 year. In [11], it is shown that geometric interpretation of fractional integration is "'Shadows on the walls"' and its Physical interpretation is "'Shadows of the past"'.

In the last years has found use in studies of viscoelastic materials, as well as in many fields of science and engineering including fluid flow, rheology, diffusive transport, electerical networks, electromagnetic theory and probability.

In this paper we consider different definitions of fractional derivatives and integrals (differintegrals). For some elementary functions, explicit formula of fractional drevative and integral are presented. Also we present some applications of fractional calculus in science and engineering.

2 Different Definitions

In this section we consider different definitions of fractional calculus.

1. **L. Euler(1730):**

   Euler generalized the formula

   $$\frac{d^n x^m}{dx^n} = m(m-1)\cdots(m-n+1)x^{m-n}$$

   by using of the following property of Gamma function,

   $$\Gamma(m+1) = m(m-1)\cdots(m-n+1)\Gamma(m-n+1)$$

   to obtain

   $$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}.$$

   Gamma function is defined as follows.

   $$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \qquad Re(z) > 0$$

2. **J. B. J. Fourier (1820 - 1822):**

   By means of integral representation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz)dp$$

he wrote

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz + n\frac{\pi}{2})dp,$$

3. **N. H. Abel (1823- 1826):**

   Abel considered the integral representation $\int_0^x \frac{s'(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x)$ for arbitrary α and then wrote

$$s(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}}.$$

4. **J. Lioville (1832 - 1855):**

   I. In his first definition, according to exponential representation of a function $f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$, he generalized the formula $\frac{d^m e^{ax}}{dx^n} = a^m e^{ax}$ as

$$\frac{d^\nu f(x)}{dx^\nu} = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n x}$$

   II. Second type of his definition was *Fractional Integral*

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)}\int_0^\infty \Phi(x+\alpha)\alpha^{\mu-1}d\alpha$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)}\int_0^\infty \Phi(x-\alpha)\alpha^{\mu-1}d\alpha$$

   By substituting of $\tau = x + \alpha$ and $\tau = x - \alpha$ in the above formulas respectively, he obtained

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)}\int_x^\infty (\tau - x)^{\mu-1}\Phi(\tau)d\tau$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)}\int_{-\infty}^x (x-\tau)^{\mu-1}\Phi(\tau)d\tau.$$

III. Third definition, includes *Fractional derivative*,

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{(-1)^\mu}{h^\mu}\Big(F(x)\frac{\mu}{1}F(x+h)++\frac{\mu(\mu-1)}{1\cdot 2}F(x+2h)-\cdots\Big)$$

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{1}{h^\mu}\Big(F(x)\frac{\mu}{1}F(x-h)++\frac{\mu(\mu-1)}{1\cdot 2}F(x-2h)-\cdots\Big).$$

5. **G. F. B. Riemann (1847 - 1876):**

   His definition of Fractional Integral is

$$D^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{\nu-1}f(t)dt + \psi(t)$$

6. **N. Ya. Sonin (1869), A. V. Letnikov (1872), H. Laurent (1884), N. Nekrasove (1888), K. Nishimoto (1987-):**

   They considered to the Cauchy Integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_c \frac{f(t)}{(t-z)^{n+1}}dt$$

   and substituted n by ν to obtain

$$D^\nu f(z) = \frac{\Gamma(\nu+1)}{2\pi i}\int_c^{x^+} \frac{f(t)}{(t-z)^{\nu+1}}dt.$$

7. **Riemann-Liouvill definition:**

   The popular definition of fractional calculus is this which shows joining of two previous definitions.

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$
$$(n-1\leq\alpha<n)$$

8. **Grünwald-Letnikove:**

   This is another joined definition which is sometimes useful.

$${}_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j}f(t-jh)$$

9. **M. Caputo (1967):**

   The second popular definition is

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}} \ , \quad (n-1 \le \alpha < n) $$

10. **K. S. Miller, B. Ross (1993):**

    They used differential operator D as

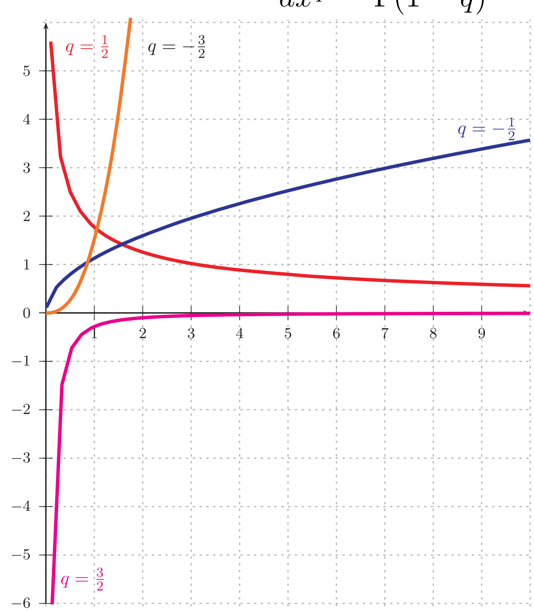
$$ D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_n} f(t), \quad \bar{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n) $$

which D^{α_i} is Riemann-Liouvill or Caputo definitions.

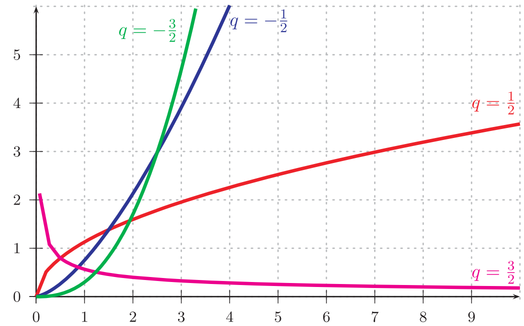
3 Fractional derivative of Some special Functions

In this section we give more explicit formulas of fractional derivative and integral of some special functions and then consider to there graph.

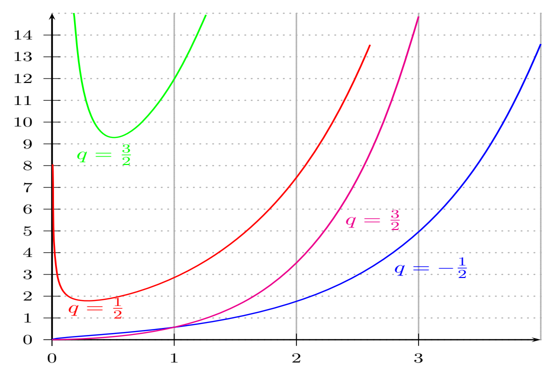
1. **Unit function:** For $f(x) = 1$ we have $\dfrac{d^q 1}{dx^q} = \dfrac{x^{-q}}{\Gamma(1-q)}$ for all q



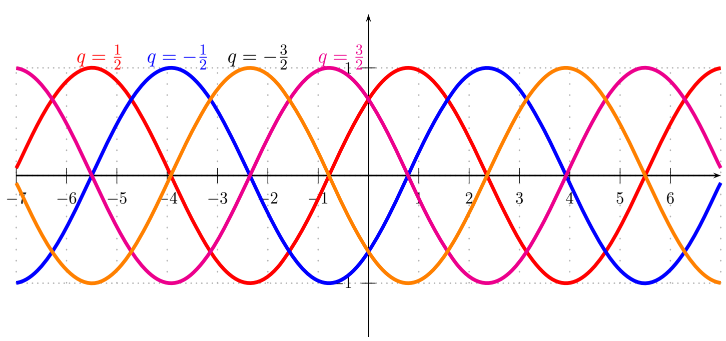
2. **Identity function:** For $f(x) = x$ we have $\dfrac{d^q x}{dx^q} = \dfrac{x^{1-q}}{\Gamma(2-q)}$



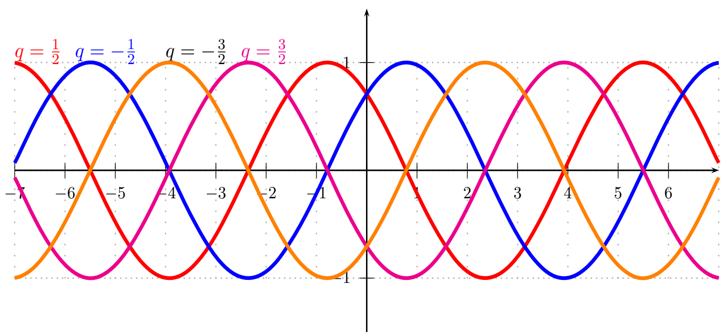
3. **Exponential function:** Fractional differintegral of the function $f(x) = e^x$ is $\dfrac{d^q e^{\pm x}}{dx^q} = \displaystyle\sum_{k=0}^{\infty} \frac{x^{k-q}}{\Gamma(k-q+1)}$



4. **Sin function:** If $f(x) = \sin x$ then $\dfrac{d^q \sin(x)}{dx^q} = \sin\left(x + \dfrac{q\pi}{2}\right)$



5. **Cosin function:** If $f(x) = \cos x$ then $\dfrac{d^q \cos(x)}{dx^q} = \cos\left(x + \dfrac{q\pi}{2}\right)$



4 Applications of Fractional Calculus

The basic mathematical ideas of fractional calculus (integral and differential operations of noninteger order)were developed long ago by the mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others and brought to the attention of the engineering world by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by Oldham and Spanier. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics. Here we state some of applications.

1. **First one**
   It may be important to point out that the first application of fractional calculus was made by Abel(1802-1829) in the solution of an integral equation that arises in the formulation of the *tautochronous problem*. This problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass m can fall in a time that is independent of the starting position. If the sliding time is constant T, then the Abel integral equation(1823) is

$$\sqrt{2g}T = \int_0^{\eta} (\eta - y)^{-\frac{1}{2}} f'(y)dy,$$

   where g is the acceleration due to gravity, (ξ, η) is the initial position and $s = f(y)$ is the equation of the sliding curve. It turns out that this equation is equivalent to the fractional integral equation

$$T\sqrt{2g} = \Gamma(\frac{1}{2})_0 D_{\eta}^{-\frac{1}{2}} f'(\eta)$$

   Indeed, Heaviside gave an interpretation of $\sqrt{p} = D^{\frac{1}{2}}$ so that ${}_0D_t^{\frac{1}{2}}1 = \frac{1}{\sqrt{\pi t}}$.

2. **Electric transmission lines**
During the last decades of the nineteenth century, Heaviside successfully developed his operational calculus without rigorous mathematical arguments. In 1892 he introduced the idea of fractional derivatives in his study of electric transmission lines. Based on the symbolic operator form solution of heat equation due to Gregory(1846), Heaviside introduced the letter p for the differential operator $\frac{d}{dt}$ and gave the solution of the diffusion equation
$$\frac{\partial^2 u}{\partial x^2} = a^2 p$$
for the temperature distribution $u(x,t)$ in the symbolic form
$$u(x,t) = A \exp(ax\sqrt{p}) + B \exp(-ax\sqrt{p})$$
in which $p \equiv \frac{d}{dx}$ was treated as constant, where a, A and B are also constant.

3. **Ultrasonic wave propagation in human cancellous bone**
N. Sebaa, Z. E. A. Fellah, W. Lauriks, C. Depollier[12]

Fractional calculus is used to describe the viscous interactions between fluid and solid structure. Reflection and transmission scattering operators are derived for a slab of cancellous bone in the elastic frame using Blot's theory. Experimental results are compared with theoretical predictions for slow and fast waves transmitted through human cancellous bone samples

4. **Modeling of speech signals using fractional calculus**
Assaleh, K.; Ahmad, W.M.[1]

In this paper, a novel approach for speech signal modeling using fractional calculus is presented. This approach is contrasted with the celebrated Linear Predictive Coding (LPC) approach which is based on integer order models. It is demonstrated via numerical simulations that by using a few integrals of fractional orders as basis functions, the speech signal can be modeled accurately.

5. **Modeling the Cardiac Tissue Electrode Interface Using Fractional Calculus**
R.L. Magin [7]

The tissue electrode interface is common to all forms of biopotential recording (e.g., ECG, EMG, EEG) and functional electrical stimulation (e.g., pacemaker, cochlear implant, deep brain stimulation). Conventional lumped element circuit models of electrodes can be extended by generalization of the order of differentiation through modification of the defining current-voltage relationships. Such fractional order models provide an improved description of observed bioelectrode behaviour, but recent experimental studies of cardiac tissue suggest that additional mathematical tools may be needed to describe this complex system.

6. **Application of Fractional Calculus to the sound Waves Propagation in Rigid Porous Materials**
   Z. E. A. Fellah, C.Depollier[3]

   The observation that the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behaviour of sound waves in this kind of materials, including relaxation and frequency dependence.

7. **Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles**
   J.I. Suárez , B.M. Vinagre , A.J. Calderón , C.A. Monje and Y.Q. Chen[14]

   Here it is presented the use of Fractional Order Controllers (FOC) applied to the path-tracking problem in an autonomous electric vehicle. A lateral dynamic model of a industrial vehicle has been taken into account to implement conventional and Fractional Order Controllers. Several control schemes with these controllers have been simulated and compared.

8. **Application of fractional calculus in the theory of viscoelasticity**
   E. Soczkiewicz[13]

   The advantage of the method of fractional derivatives in theory of viscoelasticity is that it affords possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters. Also the fractional derivative method has been used in studies of the complex moduli and impedances for various models of viscoelastic substances.

9. **Fractional differentiation for edge detection**
   B. Mathieu, P. Melchior, A. Oustaloup, Ch. Ceyral[9]

   In image processing, edge detection often makes use of integer-order differentiation operators, especially order 1 used by the gradient and order 2 by the Laplacian. This paper demonstrates how introducing an edge detector based on non-integer (fractional) differentiation can improve the criterion of thin detection, or detection selectivity in the case of parabolic luminance transitions, and the criterion of immunity to noise, which can be interpreted in term of robustness to noise in general.

10. **Wave propagation in viscoelastic horns using a fractional calculus rheology model**
Margulies, Timothy[8]

The complex mechanical behavior of materials are characterized by fluid and solid models with fractional calculus differentials to relate stress and strain fields. Fractional derivatives have been shown to describe the viscoelastic stress from polymer chain theory for molecular solutions. Here the propagation of infinitesimal waves in one dimensional horns with a small cross-sectional area change along the longitudinal axis are examined. In particular, the linear, conical, exponential, and catenoidal shapes are studied. The wave amplitudes versus frequency are solved analytically and predicted with mathematical computation. Fractional rheology data from Bagley are incorporated in the simulations. Classical elastic and fluid "Webster equations" are recovered in the appropriate limits. Horns with real materials that employ fractional calculus representations can be modeled to examine design trade-offs for engineering or for scientific application.

11. **Application of Fractional Calculus to Fluid Mechanics**
Vladimir V. Kulish and José L. Lage[4]

Application of fractional calculus to the solution of time-dependent, viscous-diffusion fluid mechanics problems are presented. Together with the Laplace transform method, the application of fractional calculus to the classical transient viscous-diffusion equation in a semi-infinite space is shown to yield explicit analytical (fractional) solutions for the shear-stress and fluid speed anywhere in the domain. Comparing the fractional results for boundary shear-stress and fluid speed to the existing analytical results for the first and second Stokes problems, the fractional methodology is validated and shown to be much simpler and more powerful than existing techniques.

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