Economic Optimization Process;
Integer Linear Programming;
Geometric Maximum Probability Problems

A. A. Arnao 1, G. Caristi 2, F. Freni 3 and A. Puglisi 1

1 IRIB-CNR Messina, Italy
2 Department of Economics, University of Messina, Italy
3 Pegaso Telematic University, Italy

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2022 Hikari Ltd.

Abstract

The mathematical approach of Linear Programming is essential to address and solve a very large number of real problems, typical of the applied sciences, when there is an indivisibility of the good to be produced or of the resource to be used and thus the consequent need to represent the problem through models of Linear Programming with integer variables. Integer Linear Programming (ILP) deals in a systematic way with the minimization (maximization) problem of a linear function with several variables (some also of a random nature), subject to equality constraints and linear inequality and to the constraint that one or more variables has integer values. The applications of the method concern many applications of modern society and extend from industrial analysis for the distribution of goods and the sequencing of productive activities, to economic problems aimed at the optimal management of a securities portfolio, to those of planning and optimal planning of public investments, to the problems inherent in biology, high energy physics, engineering and robotics [15]. In this paper, considering a new classical Laplace problems for a regular lattice, we determine the probability of intersection of a body test. In particular, we determine the values which the probability assumes the maximum value reducing the probability interval and simplifying the decision maker’s task.

Mathematics Subject Classification: 53C65; 52A22
Keywords: Geometric probability; Laplace problem; Integrated Linear Programming, Maximum value of probability, Economics optimization problems

1 Introduction

Through specific geometric probability techniques elaborated below, we tried to provide the decision maker with the formal basis of the strategic choice aimed at optimizing the overall results of the action under consideration, taking into account both the different constraints posed by the problem, and the context scenario, proceeding from a Life-Cycle Cost Analysis perspective. Project Management and optimization processes choices to maximize resource allocation results. Through specific geometric probability techniques elaborated below, we tried to provide the decision maker with the formal basis of the strategic choice aimed at optimizing the overall results of the action under consideration, taking into account both the different constraints posed by the problem, and the context scenario, proceeding from a Life-Cycle Cost Analysis perspective [15]. In this work a mathematical methodology is developed that refers to the geometric analysis of the variables in a suitable $R^m$ space, with a polyhedral approach, useful for optimizing strategic choices in public decision-making processes, i.e. in corporate ones. The criterion also lends itself to being applied in the various sectors of the business economy, also to solve problems of resource allocation (investments, allocation of personnel to strategic areas, etc.) [7]. In practice, it is often necessary to tackle problems in which the decision variables represent indivisible quantities (such as the number of trains on a relationship OD of a railway network, of people assigned to specific company tasks, etc.) or others characterized by the need to choose between a finite number of different alternatives. In the latter case, in particular, Linear Programming problems 0-1 will arise, ie problems in which the variables are binary and take the value 0 or 1. The methodology proposed here allows the construction of a mathematical model based on PLI integrated linear programming and studies the so-called ”cutting planes” in $R^m$: the resolution of the problem is carried out by eliminating the ”integrity” constraint of the variables. If the solution found at a generic step of the process is continuous, then it is also excellent for the original problem with integer variables; otherwise, the method continues by generating characteristic cutting planes, or new inequalities to add to the relaxed problem, which must render the current solution inadmissible and not discard any of the solutions of the original problem in the study (see fig. 1)
The problems of the cutting plan are increasingly important in these economic-productive sectors, because they allow an appropriate choice of the truncation plan in each iteration of the whole Linear Programming algorithm, ensuring in the mathematical treatment of the specific cases-study the rapidity of convergence of the characteristic functions. Several studies have been done on the matter, based on the study of the feasible cutting planes polynomial structures. Ralph Edward Gomory [9] proved that the optimal cutting plane is the one that maximizes the number of feasible integer points the cut touches. A theorem introduced by Georg Alexander Pick [10] allows calculating the area of each polygon whose vertices belong to a bi-dimensional lattice, as a function of the number of its internal and boundary lattice points. In 1957, John Edmund Reeve proposed a generalization of Pick’s theorem to the three dimensional case [12]. Many empirical problems, such as those typical of the economic-financial sector, or in studies of investment optimization, or for the optimal choice of alternatives for the construction of network infrastructures (tangible and intangible), can be represented by means of particular programming models linear integer, in which the objective function and the constraints are integer variables. Motivated by above facts in the present paper starting from results obtained in recent years [1], [2],..., [5] we consider one cutting plane problem for a regular lattice (see in fig. 2). Our goal is to determine the maximum value of the probability in order to help the decision maker’s task.
2 Preliminaries

In this section we present some results and considerations that will be needed in the rest of the paper.

Consider the regular lattice $\mathcal{R}(a, b; \alpha)$ with a fundamental region $C_0$ composed by trapezoid isosceles (fig. 2):

![Fig.2](image)

where $a < b$ and $\frac{\pi}{4} < \frac{\pi}{2}$.

We know that, any congruent polygon can be inlaid in a plane. In this way we obtain a lattice that covers the plane. A set of points in the plane is called a domain if it is open and connected. A set of points is called a region if it is the union of a domain with some, or all of its boundary points. From the lattice of fundamental regions in the plane, we understand a sequence of congruent regions that represent the Santalò conditions [13]. We want to compute the probability that a segment $s$ with random position and constant length $l < 1/2$ intersects a side of lattice $\mathcal{R}$, i.e. the probability $P_{int}$ that a segment $s$ intersects a side of the fundamental cell $C_0$.

The position of the rectangle $r$ is determined by its middle point and by the angle $\varphi$ that the side of length $m$ formed with the line $BC \circ AD$.

Fengfan and Deyi [8] study similar problem using two concepts, the generalized support function and restricted chord function, both referring to the convex set, which were introduced by Delin in [9].

To compute the probability $P_{int}$ we consider the limiting positions of segment $s$, for a specified value of $\varphi$, in the cells $C_0$, (fig.3).
By denoting $M_i (i = 1, ..., 3)$ as the set of segment $s$ which intersect a side of the cell $C_{0i}$ and with $N_i$ the set of segment $s$ all contained in the cell $C_{0i}$ we have [14]:

$$P_{int} = 1 - \sum_{i=1}^{3} \frac{\mu(N_i)}{\mu(M_i)},$$

(1)

where $\mu$ is the Lebesgue measure in the Euclidean plane.

To compute the above measure $\mu(M_i)$ and $\mu(N_i)$ we use the Poincaré kinematic measure [11] $dk = dx \wedge dy \wedge d\varphi$, where $x, y$ are the coordinates of the middle point of $s$ and $\varphi$ is the fixed angle.

### 3 Main results

We can prove

**Theorem 1** The probability that a random segment $s$ intersects a side of lattice $\mathbb{R}$ is:

$$P_{int} = \frac{1}{(b^2 - a^2) \alpha \tan \alpha} \left[ \frac{b(\cos \alpha - \cos 2\alpha) - a(1 - \cos \alpha)}{\cos \alpha} - \frac{l}{2} (1 - \cos 2\alpha) \right].$$

**Proof.** By Fig. 2 we have

$$|AB| = |CD| = \frac{b - a}{2 \cos \alpha}$$

(2)

and

$$\text{area} C_0 = \frac{b^2 - a^2}{2} \tg \alpha.$$  

(3)

Since $\varphi \in [0, \alpha]$ we can write

$$\mu(M) = \int_{0}^{\alpha} d\varphi \int_{\{(x,y) \in C_0\}} dx dy = \int_{0}^{\alpha} (\text{area} C_0) d\varphi = \alpha \text{area} C_0,$$

(4)
and
\[ \mu(N) = \int_0^\alpha d\varphi \int\int_{\{(x,y)\in\tilde{C}_0\}} \{ x, y \} \varphi \cdot dx \cdot dy = \int_0^\alpha \{ \text{area} \tilde{C}_0(\varphi) \} d\varphi = (5) \]
\[ \frac{l}{2\cos\alpha} \{ b(\cos\alpha - \cos 2\alpha) - a(1 - \cos\alpha) \} - \frac{l^2}{4} (1 - \cos 2\alpha). \]

Computing (4) and (5) and considering (1) and (3) we obtain the probability (2). ■

Remark 2 Considering
\[ f(\alpha) = \frac{1}{\alpha \sin\alpha} \left[ b(\cos\alpha - \cos 2\alpha) - a(1 - \cos\alpha) - \frac{l}{2} \cos\alpha (1 - \cos 2\alpha) \right], \]
we have
\[ f'(\alpha) = \frac{1}{\alpha^2 \sin^2\alpha} \{ \alpha \sin\alpha \left[ b(2\sin 2\alpha - \sin\alpha) - a\sin\alpha - \frac{l}{2} \cos\alpha (1 - \cos 2\alpha + 2\cos\alpha \sin\alpha - \sin\alpha) \right] - \\
(\sin\alpha + \alpha \cos\alpha) \left[ b(\cos\alpha - \cos 2\alpha) - a(1 - \cos\alpha) - \frac{l}{2} \cos\alpha (1 - \cos 2\alpha) \right] \}. \]

For \( \alpha = \pi/4 \) and
\[ 2a\left[ 2\sqrt{2} + 2\pi \left( \sqrt{2} - 1 \right) - 8 \right] - b\left[ 4\pi - \sqrt{2}(\pi + 4) + \sqrt{2}(2 + \pi) \right] l = 0 \]
we have
\[ f'(\alpha) = 0. \]

Easily we have that for these values
\[ f''(\alpha) < 0, \]
and then
\[ P_{\text{int}} = \frac{l}{2(b^2 - a^2)} \cdot f(\alpha) \]
is maximum.

4 Conclusion

For the identification of the variables in the strategic planning of public investments one of the fundamental tools is represented by the SWOT (Strengths, Weaknesses, Opportunities, Threats) analysis. This evaluation technique is essential for evaluating previously generated and allows to fully examine a public or corporate initiative by the characterization of the problem both with endogenous variables and exogenous variables. These variables influence the behavior of the "system" as in the following Fig. 4.
The development of the strategy must then be studied, in order to establish the consequent "actions" and take the best decisions, according to an appropriate risk analysis approach, according to the following "logical scheme" structured in 5 successive phases (see Fig. 5).

For the systematic assessment of the risks, one can proceed, through a system of variables identified analysis for the definition of the problem and the constraints imposed by it (economic-financial, implementation times, etc.) and an assessment of the probability of occurrence (p) of the "events". These events potential risks for the objectives set of the "decision maker" and of the IMPACT (s) that they can produce in the entire reference "context". If the risks are "n" in number, by associating each of them with a "weight" $w_i$ ($i = 1, ..., n$) the problem can be schematized through the mathematical relationship:

$$R = \sum_{i}^{n} [R_i = f (p_i \cdot i_i)]^{-w_i}, \quad \text{with} \quad \sum w_i = 1.$$
For a qualitative analysis of the "global effects", reference can then be made to special "matrix" such as the one reproduced below as an example in the Cartesian plan and characterized by "risk levels" defined in a chromatic way according to "severity".

![Risk Analysis Matrix]

Fig. 6

It should be emphasized how the Risk Analysis can be useful to the "decision maker" to pursue the following strategic objectives:
- Predict and reduce the negative effects of one or more "adverse events";
- Evaluate whether (and how) the potential "risk factors" related to the specific initiative in question are sufficiently "counterbalanced" to the ensemble of "usefulness" deriving from it; in this case the analysis can be very important (if not decisive) within the decision-making process for the continuation of the action or its "remodeling" or even a "rethinking";
- Identify which and how many resources would be needed in the event that the "risk" materializes and the "impacts" on the budget;
- To envisage "sensitive" changes within the so-called "vast area" and / or the sector in which public (or private) intervention is located and to prepare accordingly.

For example, in the business field, the analysis can highlight a future scenario in which the market area can see the entry of new "competitors", or the same is subject to new regulatory developments by the legislator, or still other potential scenarios.

References

https://doi.org/10.12988/ams.2014.411916
Economic optimization process


Received: February 21, 2022; Published: March 19, 2022