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On Approximating Expected Log-Utility of Mutual Fund Investments

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Abstract

In this paper, considering the utility measured as a logarithmic function associated with rate of return, by examining historical data of a set of mutual funds, we study different approximating methods of the expected utility and compare their related approximation errors.

Keywords: Log-utility, Expected Value, Approximation

1. Introduction

Utility function is an important economics concept. It is a mathematical way to represent satisfactions or preferences. One general restriction is that the utility function, defined on real number and real value, be increasing, continues and concave. One commonly used utility measure is log-utility $U(x) = \ln(1 + x)$.

2. Approximation Methods

In this section, we will approximate utility,EU, using a function of mean and variance. All these three methods had been introduced in [1] and [2].
Let $e = E(r), v = Var(r), \sigma = \sqrt{v}$, where r is the rate of return.

A) Mid-point approximation

$$E[U(r)] \cong \frac{U(e-\sigma) + U(e+\sigma)}{2} \quad (1)$$

B) Taylor series approximation

Markowitz [1] introduced a method to approximate EU based on Taylor-series around e:

$$U(r) = U(e) + U'(e)(r-e) + .5U''(e)(r-e)^2 + \cdots$$

Since $e = E(r), v = E[(r-e)^2]$, then

$$E[U(r)] \cong U(e) + .5U"(e)v \qquad (2)$$

C) Three points quadratic approximation

Levy and Markowitz [2] introduced an 'alternate way' class of estimating functions that were selected in which the quadratic was fit to three points:

$$\big(e-k\sigma,\ U(e-k\sigma)\big),\ \big(e, U(e)\big),\ (e+k\sigma, U(e+k\sigma))$$

Write the quadratic functions as:

$$Q_k(r) = a_k + b_k(r-e) + c_k(r-e)^2$$

Then,

$$E[Q_k(r)] = a_k + c_k v$$

and

$$a_k = U(e)$$

$$b_k = \frac{U(e+k\sigma) - U(e-k\sigma)}{2k\sigma}$$

$$c_k = \frac{U(e+k\sigma) + U(e-k\sigma) - 2U(e)}{2k^2\sigma^2}$$

From that we got

$$E[Q_k(r)] = U(e) + \frac{U(e+k\sigma)+U(e-k\sigma)-2U(e)}{2k^2}$$

Clearly, if $k = 1$, then $E[Q_1(r)] = \frac{U(e+\sigma)+U(e-\sigma)}{2}$, which is same as the mid-point estimation in equation (1).

In fact, Levy and Markowitz [2] also pointed out if $k \to 0$, then $E[Q_{k\to 0}(r)] \to U(e) + .5U"(e)v$, same as Tylor series approximation in equation (3).

For readers' convenience, we provide a proof of $k \to 0$ result as follows.
Let $h = k\sigma$. Since σ is a finite constant,

$$
\begin{aligned}
&\lim_{k\to 0}\frac{U(e+k\sigma)+U(e-k\sigma)-2U(e)}{k^2\sigma^2}\\
&=\lim_{h\to 0}\frac{U(e+h)+U(e-h)-2U(e)}{h^2}\\
&=\lim_{h\to 0}\frac{\frac{U(e+h)-U(e)}{h}-\frac{U(e)-U(e-h)}{h}}{h}\\
&=\lim_{h\to 0}\frac{U'(e)-U'(e-h)}{h}\\
&=U''(e)
\end{aligned}
$$

Of course, people can also take the second derivative of the numerator and denominator on the first step with respect to ‘k’ and then apply L’Hospital’s rule.

3. Empirical Results

If we choose $k = 0.5$, then

$$E[Q_{0.5}(r)] \cong 2[U(e+0.5\sigma)+U(e-0.5\sigma)]-3U(e). \tag{3}$$

For monthly return of ten mutual funds from Jan. 2011 to Jan. 2017, Table 1 provides results of these three estimations stated in equations (1), (2) and (3), for $U(r) = \ln(1+r)$.

Name	$E(r)$	$U[E(r)]$	$E[U(r)]$	$Q_1(r)$	$Q_{0.5}(r)$	$Q_0(r)$
ACMTX	-0.00283526	-0.00283928	-0.00372513	-0.00373770	-0.00373710	-0.00373690
AAAAX	0.00082919	0.00082885	0.00064179	0.00063992	0.00063994	0.00063995
AAXAX	-0.00102976	-0.00103029	-0.00121315	-0.00121557	-0.00121554	-0.00121553
UTAYX	0.00406229	0.00405406	0.00400727	0.00400695	0.00400695	0.00400695
ATHA	0.00787630	0.00784545	0.00701957	0.00701346	0.00701398	0.00701410
MQIFX	0.00680535	0.00678230	0.00644868	0.00644859	0.00644867	0.00644869

GRECX	0.00824170	0.00820792	0.00724135	0.00722475	0.00722548	0.00722565
BIPSX	0.02437113	0.02407889	0.02055993	0.02064221	0.02065106	0.02065204
UDPIX	0.01670866	0.01657061	0.01451724	0.01449520	0.01449843	0.01449894
AFBIX	-0.01002325	-0.01007382	-0.01028981	-0.01029111	-0.01029107	-0.01029108

Table 1: $U(r) = \ln(1 + r)$, r is the monthly return rate.

Table 1 shows that all three methods $(Q_0, Q_{0.5}, Q_1)$ presented here can be considered very good approximations of $E(U(r))$. In fact, the correlation coefficients (easily calculated by using 'CORREL' function in Excel) between $E[U(r)]$ and each of $Q_0, Q_{0.5}, Q_1$ are over 0.99. It may also be noticed that column values in $U[E(r)]$ are greater than values in $E[U(r)]$, which matches Jensen's inequality of $E[U(r)] \leq U[E(r)]$ for $U(r) = \ln(1 + r)$ is a concave function.

For reader's convenience, steps of obtaining Table 1 in Excel are also provided as follows:

1. Download and import monthly data of these 10 mutual funds from Jan. 2011 to Jan. 2017. A good data resource is yahoo finance webpage.
2. Evaluate monthly returns 'r' of each mutual fund.
3. Evaluate average monthly return, simple mean '$e = E(r)$' and simple variance '$VAR(r)$' of each mutual fund.
4. Evaluate '$U[E(r)]$'by calculating '$LN(1 + e)$'.
5. Evaluate '$E[U(r)]$' by calculating the average of '$LN(r)$'.
6. Evaluate '$Q_0(r)$', '$Q_{0.5}(r)$'and '$Q_1(r)$'by calculating
   $Q_1(r) = \frac{LN(1+e-\sigma)+LN(1+e+\sigma)}{2}$;
   $Q_{0.5}(r) = 2[LN(1 + e - 0.5\sigma) + LN(1 + e + 0.5\sigma)] - 3LN(e)$;
   $Q_0(r) = LN(1 + e) - \frac{1}{2} \cdot \frac{1}{(1+LN(1+e))^2} VAR(r)$;
   Here, $e = E(r)$ and $\sigma = SQRT(VAR(r))$.

4. Further Disscussions

In this paper, we used mutual funds data and log-utility to work on the approximation. It can also be worked on other types of financial data like price of stocks, options, or hedge funds [3] and other types of utility functions like exponential utility $U(x) = -e^{-ax}$for some $a > 0$ or power utility $U(x) = bx^b$ for some $b < 1$. It is a good example to show how numerical analysis applied to economics and finance.

For readers with advanced numerical analysis skills, they may want to test other type of approximations. For example, Chebyshev series estimation, which is one kind of orthogonal polynomial approximations, states that

$$U(r) = \sum_{n=0}^{\infty} a_n T_n(r)$$

where the Chebyshev polynomials of the first kind are defined by the recurrence relation:

$$T_0(r) = 1$$
$$T_1(r) = r$$
$$T_{n+1}(r) = 2rT_n(r) - T_{n-1}(r)$$

If $U(r) = \ln(1 + r)$, then $a_n = \begin{cases} -\ln 2\,, & n = 0 \\ \frac{-\pi(-1)^n}{n}, & n > 0 \end{cases}$.

Interested readers may want to check on this approximation method.

References

[1] Harry Markowitz, *Portfolio Selection: Efficient Diversification of Investment*, Yale University, 1971.

[2]. H. Levy and H. M. Markowitz, Approximating Expected Utility by a Function of Mean and Variance, *The American Economic Review*, **69** (1979), no. 3, 308-317.

[3] W. Fung and D. Hsieh, Is mean-variance analysis applicable to hedge funds?, *Economics Letters*, **62** (1999), no. 1, 53-58.
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