Non-tangential Maximal Moduli and Harmonic Functions on the Upper Half Plane (Part 1)

Hon Ming Ho and Kin Y. Li

Department of Mathematics
Hong Kong University of Science and Technology, Hong Kong

Abstract

In this paper, we present some nice theorems on non-tangential maximal moduli associated to real-valued harmonic functions on the upper half plane. Some necessary and sufficient conditions using non-tangential moduli to characterize real parts of holomorphic functions with their boundary value functions having smoothness of high order will be presented in this paper and the second paper.

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1 Introduction

Let \( \mathbb{C}, \mathbb{R} \) and \( \mathbb{N} \) denote the sets of all complex numbers, real numbers and positive integers respectively. In this paper, we will consider some harmonic functions defined on \( \mathbb{R}^2_+ = \{(x, y) : x \in \mathbb{R}, y > 0\} \). In our previous papers [1] and [2], we studied certain modularity concepts like \( \omega^*_\infty(t; u) \) and \( \tau^*_\infty(t; u) \) which are some types of moduli associated to \( u \) that can be used to characterize those harmonic functions defined on the open unit disk \( U \) having continuous and Riemann integrable boundary value functions.

We ask the following question: if we have \( \omega^*_p(t; u) = O(t^r) \) as \( t \to 0^+ \), then can we have boundary value function of \( u \) with higher smoothness property (e.g. differentiable almost everywhere)? We know that if \( u \) is a real-valued harmonic function defined on upper half plane \( \mathbb{R}^2_+ \), then \( u \) is the real part of
some holomorphic function $F$ in the Hardy space $H^p(\mathbb{R}^2)$ with $p \in (0, +\infty)$
if and only if its nontangential maximal function $Nu$ of $u$ belongs to $L^p(\mathbb{R}^1)$. We also know that the boundary value functions of holomorphic functions in $H^p(\mathbb{R}^2)$ are $L^p$-functions on $\mathbb{R}^1$ where $p \in [1, +\infty]$. Then it is natural to ask the following question: if $u$ is a real-valued harmonic function defined on $\mathbb{R}^2$, then can we have some kinds of moduli associated to $u$ which can be used to characterize $u$, the real part of some holomorphic function $F$ with its boundary value function having smoothness of higher order $p \in [1, +\infty]$? In turn, this question motivated us to introduce the notion of “non-tangential maximal moduli”. Now we begin with some definitions and facts.

**Definition 1.1** Let $M_1, M_2, M_3, \ldots$ be defined inductively on $\mathbb{R}$ by

$$M_1(x) = \chi_{(0,1)}(x), \quad M_{j+1}(x) = (M_j * M_1)(x) = \int_{\mathbb{R}} M_j(x - \omega)M_1(\omega) \, d\omega,$$

where $\chi_{(0,1)}$ is the characteristic function of the interval $(0,1)$.

**Lemma 1.2** Each $M_j$ is a nonnegative function with support $(0, j)$. Moreover, each $M_j$ is bounded by 1 with integral equal to 1. Here, “support on $(0, j)$” means $\{x \in \mathbb{R} : M_j(x) \neq 0\} \subseteq (0, j)$.

**Proof.** Let $P(n)$ be the proposition that $M_n$ is nonnegative, supported on $(0, n)$, bounded by 1 with integral equal to 1. When $n = 1$, this is obvious. Assume that $P(n)$ is true, where $n \geq 1$. For every $x \in \mathbb{R}$,

$$M_{n+1}(x) = \int_{\mathbb{R}} M_n(x - \omega)M_1(\omega) \, d\omega = \int_0^1 M_n(x - \omega) \, d\omega.$$

If $x \notin (0, n+1)$ and $\omega \in [0, 1]$, then there are 2 cases. In case 1, if $x \leq 0$, then $x - \omega \leq -\omega \leq 0$ and in case 2, if $n + 1 \leq x$, then $n + \omega \leq n + 1 \leq x$, i.e. $n \leq x - \omega$. Combine these 2 cases, we have $x - \omega \notin (0, n)$. Then $M_n(x - \omega) = 0$. Therefore, $M_{n+1}$ is nonnegative, supported on $(0, n+1)$ and bounded by 1. Moreover, since $\int_{\mathbb{R}} M_n(\xi) \, d\xi = 1$, by Fubini’s theorem,

$$\int_{\mathbb{R}} M_{n+1}(x) \, dx = \int_{\mathbb{R}} \int_0^1 M_n(x - \omega) \, d\omega \, dx = \int_0^1 \int_{\mathbb{R}} M_n(x - \omega) \, dx \, d\omega = 1.$$

Then $P(n)$ implies $P(n+1)$ for every positive integer $n$. By mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Definition 1.3** For $f : \mathbb{R} \to \mathbb{C}$, $h \in \mathbb{R}$ and $n \in \mathbb{N}$, define $(\Delta^1_h f)(x) = f(x + h) - f(x)$ for every $x \in \mathbb{R}$ and $\Delta^{n+1}_h f = \Delta^n_h(\Delta^1_h f)$. 


Lemma 1.4 Let \( n \in \mathbb{N}, f \in C^n(\mathbb{R}), h \in \mathbb{R} \setminus \{0\} \), where
\[
C^n(\mathbb{R}) = \{ f : f \text{ has continuous derivative on } \mathbb{R} \text{ up to order } n \}.
\]
Then for every \( x \in \mathbb{R} \), \( (\Delta_h^n f)(x) = \int_{\mathbb{R}} f^{(n)}(x + h\xi)h^n M_n(\xi) \, d\xi \).

**Proof.** Let \( P(n) \) be the proposition that for all \( f \in C^n(\mathbb{R}), h \in \mathbb{R} \setminus \{0\}, x \in \mathbb{R} \),
\[
(\Delta_h^n f)(x) = \int_{\mathbb{R}} f^{(n)}(x + h\xi)h^n M_n(\xi) \, d\xi.
\]
For \( n = 1, h \in \mathbb{R} \setminus \{0\}, x \in \mathbb{R}, f \in C^1(\mathbb{R}) \), as \( \chi_{(x,x+h)}(\xi) = M_1(\frac{\xi-x}{h}) \), we have
\[
(\Delta_h^1 f)(x) = f(x + h) - f(x) = \int_x^{x+h} f^{(1)}(\xi) \, d\xi = \int_{\mathbb{R}} f^{(1)}(\xi)\chi_{(x,x+h)}(\xi) \, d\xi
\]
\[
= \int_{\mathbb{R}} f^{(1)}(\xi)M_1(\frac{\xi-x}{h}) \, d\xi = \int_{\mathbb{R}} f^{(1)}(x + h\omega)hM_1(\omega) \, d\omega.
\]
So \( P(1) \) is true. If \( P(n) \) is true, then let \( f \in C^{n+1}(\mathbb{R}), x \in \mathbb{R}, h \neq 0 \). So by using Fubini’s theorem, we have
\[
(\Delta_h^{n+1} f)(x) = (\Delta_h^n(\Delta_h^n f))(x) = \int_{\mathbb{R}} (\Delta_h^n(f^{(n)}))(x + h\omega)h^n M_n(\omega) d\omega,
\]
\[
= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f^{(n+1)}((x + h\omega) + h\xi)hM_1(\xi)d\xi \right] h^n M_n(\omega) d\omega,
\]
\[
= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f^{(n+1)}(x + hy)hM_1(y - \omega)dy \right] h^n M_n(\omega) d\omega, \quad (\omega + \xi = y)
\]
\[
= \int_{\mathbb{R}} f^{(n+1)}(x + hy)h^{n+1} M_{n+1}(y)dy.
\]
Then \( P(n + 1) \) is true, completing the induction.

**Corollary 1.5** Let \( n \in \mathbb{N}, x \in \mathbb{R}, f \in C^n(\mathbb{R}) \). Then \( \lim_{t \to 0} \frac{(\Delta_h^n f)(x)}{t^n} = f^{(n)}(x) \).

**Proof.** Let \( h \in \mathbb{R} \setminus \{0\} \). Since \( \int_{\mathbb{R}} M_n(\xi) d\xi = 1 \), we have
\[
\frac{1}{h^n}(\Delta_h^n f)(x) - f^{(n)}(x) = \int_{\mathbb{R}} f^{(n)}(x + h\xi)M_n(\xi) \, d\xi - \int_{\mathbb{R}} f^{(n)}(x)M_n(\xi) \, d\xi
\]
\[
= \int_{\mathbb{R}} (f^{(n)}(x + h\xi) - f^{(n)}(x))M_n(\xi) \, d\xi = \int_{0}^{h^n} (f^{(n)}(x + h\xi) - f^{(n)}(x))M_n(\xi) \, d\xi
\]
with \( \{ x \in \mathbb{R} : M_n(x) \neq 0 \} \subseteq (0, n) \). Since \( f^{(n)} \) is continuous at \( x \), for \( \varepsilon > 0 \), there is \( \delta_0 > 0 \) such that \( |\omega| < \delta_0 \) implies \( |f^{(n)}(x + \omega) - f^{(n)}(x)| < \varepsilon \). Let
\[ \delta^* = \delta_0/n \text{ (with } n \text{ fixed, } \delta_0 \text{ depends on } \varepsilon \text{ and } x) \]. Let \( t \in \mathbb{R} \setminus \{0\} \) be such that 
\( |t| < \delta^* \). For each \( \xi \in (0, n) \), \( |t\xi| < n|t| < \delta_0 \). So 
\[ \left| \frac{1}{t^n}(\Delta^n f)(x) - f^{(n)}(x) \right| \leq \frac{1}{t^n} \int_0^n |f^{(n)}(x+t\xi) - f^{(n)}(x)| M_n(\xi) \, d\xi \leq \varepsilon \int_0^n M_n(\xi) \, d\xi = \varepsilon. \]

Then the result follows.

Next we will introduce a few more useful concepts.

**Definition 1.6** Let \( p \in [1, +\infty], n \in \mathbb{N} \) and \( u : \mathbb{R}_+^2 \to \mathbb{C} \) be a function.

(a) The non-tangential maximal function of \( u \) is defined as follows: for all \( x \in \mathbb{R} \), \( (Nu)(x) = \sup_{(y,t) \in \Gamma(x)} |u(y, t)| \), where \( \Gamma(x) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : |x - \alpha| < \beta\} \).

(b) For all \( h \in \mathbb{R} \) and \( (x, y) \in \mathbb{R}_+^2 \), define \( (\Delta^1_h u)(x, y) = u(x + h, y) - u(x, y) \) and \( (\Delta^{n+1}_h u)(x, y) = (\Delta^1_h(\Delta^1_n h u))(x, y) \) for \( n = 1, 2, 3, \ldots \).

(c) For every \( t \geq 0 \), define the nontangential maximal modulus of \( u \) as follows:
\[ N_{\omega}(t; u)_p = \sup_{|h| \leq t, h \in \mathbb{R}} \|N(\Delta^1_h u)\|_{L^p(\mathbb{R})}. \]

We note that (1) in order to make \( Nu \) define properly, \( u \) must be required to be defined everywhere on \( \mathbb{R}_+^2 \) and (2) no matter what \( u \) is, \( \{x \in \mathbb{R} : (Nu)(x) > \alpha\} \) is always open. In particular, \( Nu \) is always measurable. The next three theorems are the main content of the results of this paper.

**Theorem 1.7** Let \( p \in [1, +\infty], n \in \mathbb{N} \). Let \( u : \mathbb{R}_+^2 \to \mathbb{C} \) be a function such that for all \( y > 0, u(\cdot, y) \in C^n(\mathbb{R}) \). Then \( \|N(\partial^n_x u)\|_p = \sup_{t > 0} \frac{N_{\omega}(t; u)_p}{t^n} \), where \( \partial^n_x u = \partial^n_{x_1} u \).

**Proof.** Let \( h_1, h_2, h_3, \ldots \) be a sequence of nonzero real numbers such that \( \lim_{j \to \infty} h_j = 0 \). Let \( \varepsilon_0 > 0, x_0 \in \mathbb{R}, (y, t) \in \Gamma(x_0) \). By corollary 1.5, we have
\[ \left| (\partial^n_x u)(y, t) \right|^p = \lim_{j \to \infty} \left| \frac{1}{(h_j)^n} (\Delta^n_{h_j} u)(y, t) \right|^p. \]

For some \( k_0 \), all \( j \geq k_0 \) implies \( \left| (\partial^n_x u)(y, t) \right|^p \leq \varepsilon_0 < \left| \frac{1}{(h_j)^n} (\Delta^n_{h_j} u)(y, t) \right|^p \). Then
\[ \left| (\partial^n_x u)(y, t) \right|^p \leq \varepsilon_0 < \inf_{j \geq k_0} \left| \frac{1}{(h_j)^n} (\Delta^n_{h_j} u)(y, t) \right|^p \leq \lim_{j \to \infty} \left| \frac{1}{(h_j)^n} N(\Delta^n_{h_j} u)(x_0) \right|^p. \]

Now \( \varepsilon_0, x_0 \) independent implies \( \|N(\partial^n_x u)(x_0)\|_p \leq \lim_{j \to \infty} \left| \frac{1}{(h_j)^n} N(\Delta^n_{h_j} u)(x_0) \right|^p. \)

By Fatou’s lemma,
\[ \|N(\partial^n_x u)\|_p^p \leq \lim_{j \to \infty} \left| \frac{1}{h_j} \right|^{pn} \|N(\Delta^n_{h_j} u)\|_p^p \leq \left( \sup_{t > 0} \frac{N_{\omega}(t; u)_p}{t^n} \right)^p. \]
Conversely, let \( x_0 \in \mathbb{R}, (y, t) \in \Gamma(x_0), h \in \mathbb{R} \setminus \{0\} \). By lemma 1.4, we have
\[
(\Delta^h_n u)(y, t) = \int_{\mathbb{R}} (\partial^p_n u)(y + h \xi, t) h^n M_n(\xi) \, d\xi.
\]
So
\[
\left| \frac{1}{h^n} (\Delta^h_n u)(y, t) \right| \leq \int_{\mathbb{R}} |(\partial^p_n u)(y + h \xi, t)| M_n(\xi) \, d\xi \leq \int_{\mathbb{R}} N(\partial^p_x u)(x_0 + h \xi) M_n(\xi) \, d\xi.
\]
since \( |(x_0 + h \xi) - (y + h \xi)| = |x_0 - y| < t, (y + h \xi, t) \in \Gamma(x_0 + h \xi) \). Then by Jensen inequality, lemma 1.2, Fubini’s theorem, translation invariance property of \( L^p \) norm, we get
\[
\left| \frac{1}{h^n} (\Delta^h_n u)(y, t) \right|^p \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} N(\partial^p_x u)(x + h \xi) M_n(\xi) \, d\xi \right)^p \, dx
\]
\[
\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |N(\partial^p_x u)(x + h \xi)|^p M_n(\xi) \, dx \, d\xi = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |N(\partial^p_x u)(x + h \xi)|^p \, dx \right) M_n(\xi) \, d\xi
\]
\[
= \left( \int_{\mathbb{R}} M_n(\xi) \, d\xi \right) \|N(\partial^p_x u)\|_p^p = \|N(\partial^p_x u)\|_p^p
\]
So for all \( h \in \mathbb{R}, \|N(\Delta^h_n u)\|_p \leq |h|^n \|N(\partial^p_x u)\|_p \) and \( \sup_{t>0} \frac{N\omega_n(t; u)}{t^n} \leq \|N(\partial^p_x u)\|_p \). This concludes the proof.

Now we give a version of Marchaud’s inequality for non-tangential maximal moduli. Marchaud (see [3]) proved the inequality for ordinary moduli in 1927. The idea of the proof of the theorem below is essentially the same as the idea in Marchaud’s proof. The following theorem will be used in the paper of Nontangential maximal moduli and harmonic functions on upper half plane (part 2).

**Theorem 1.8** Let \( u : \mathbb{R}^2_+ \rightarrow \mathbb{C} \) be a function. Let \( p \in [1, +\infty) \) and \( k \in \mathbb{N} \). Suppose that \( Nu \in L^p(\mathbb{R}) \). Then for all \( t > 0 \), we have
\[
N\omega_k(t; u)_p \leq k 2^k t^k \int_{\mathbb{R}} \frac{N\omega_{k+1}(s; u)_p}{s^{k+1}} \, ds.
\]

**Proof.** Define \( (T_h u)(x, y) = u(x + h, y) \) and \( (I u)(x, y) = u(x, y) \). Then
\[
(\Delta_{2h} u)(x, y) = u(x + 2h, y) - u(x, y) = (T_{2h} u)(x, y) - (I u)(x, y) = ((2h - I) u)(x, y).
\]
Moreover, \( (T_{2h} u)(x, y) = u(x + 2h, y) = (T_h (T_h u))(x, y) = (T_h^2 u)(x, y) \). Then for all \( h \in \mathbb{R} \) and \( k \in \mathbb{N} \), by using binomial theorem, we have
\[
\Delta_{2h}^k = (T_{2h} - I)^k = ((T_h + I)(T_h - I))^k = (T_h + I)^k (T_h - I)^k = (T_h + I)^k (T_h + I) \Delta_h^k
\]
\[
= \sum_{j=0}^{k} \binom{k}{j} T_h^j \Delta_h^k = \sum_{j=0}^{k} \binom{k}{j} (T_h^j - I) \Delta_h^k + \sum_{j=0}^{k} \binom{k}{j} \Delta_h^k
\]
Continuing this iteration process, we have, for every $m \in \mathbb{N}$, \[ \sum_{j=0}^{k} \binom{k}{j} \left( \sum_{\rho=0}^{j-1} T^\rho_h (T_h-I) \Delta^k_h \right) + 2^k \Delta^k_h = \sum_{j=0}^{k} \binom{k}{j} \left( \sum_{\rho=0}^{j-1} T^\rho_{ph} \Delta^{k+1}_h \right) + 2^k \Delta^k_h. \]

The last inequality follows from the fact that $T^p_h = T^{ph}_h$.

Let $x, h \in \mathbb{R}, k \in \mathbb{N}$ and $(y, t) \in \Gamma(x)$. We have

\[
(\Delta^{k}_{2h} u)(y, t) = \sum_{j=0}^{k} \binom{k}{j} \left( \sum_{\rho=0}^{j-1} (T^\rho_{ph} (\Delta^{k+1}_h u))(y, t) \right) + 2^k (\Delta^{k}_h u)(y, t),
\]

\[
(\Delta^{k}_h u)(y, t) = \frac{1}{2^k} (\Delta^{k}_{2h} u)(y, t) - \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \sum_{\rho=0}^{j-1} (T^\rho_{ph} (\Delta^{k+1}_h u))(y, t).
\]

So \[ |(\Delta^{k}_h u)(y, t)| \leq \frac{1}{2^k} |(\Delta^{k}_{2h} u)(y, t)| + \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \sum_{\rho=0}^{j-1} |(T^\rho_{ph} (\Delta^{k+1}_h u))(y, t)|, \]

\[
N(\Delta^{k}_h u)(x) \leq \frac{1}{2^k} N(\Delta^{k}_{2h} u)(x) + \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \sum_{\rho=0}^{j-1} N(\Delta^{k+1}_h u)(x + \rho h)
\]

since $(T^\rho_{ph} (\Delta^{k+1}_h u))(y, t) = (\Delta^{k+1}_h u)(y + \rho h, t)$. For $p \in [1, +\infty)$, by the translation invariant property of $L^p$ norm, we have

\[
\|N(\Delta^{k}_h u)\|_p \leq \frac{1}{2^k} \|N(\Delta^{k}_{2h} u)\|_p + \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} j \|N(\Delta^{k+1}_h u)\|_p
\]

\[
= \frac{1}{2^k} \|N(\Delta^{k}_{2h} u)\|_p + \frac{k}{2} \|N(\Delta^{k+1}_h u)\|_p.
\]

Therefore, we have

\[
N\omega_k(t; u)_p \leq \frac{1}{2^k} N\omega_{k}(2t; u)_p + \frac{k}{2} N\omega_{k+1}(t; u)_p, \quad \cdots \quad (*). \]

Replacing $t$ by $2t$, we have $N\omega_k(2t; u)_p \leq \frac{1}{2^k} N\omega_k(2^2t; u)_p + \frac{k}{2} N\omega_{k+1}(2t; u)_p$. By substituting the above inequality into $(*)$, we have

\[
N\omega_k(t; u)_p \leq \left( \frac{1}{2^k} \right)^2 N\omega_k(2^2t; u)_p + \frac{k}{2} \left( N\omega_{k+1}(t; u)_p + \frac{1}{2^k} N\omega_{k+1}(2t; u)_p \right).
\]

Continuing this iteration process, we have, for every $m \in \mathbb{N}$,

\[
N\omega_k(t; u)_p \leq \left( \frac{1}{2^k} \right)^m N\omega_k(2^m t; u)_p + \frac{k}{2} \sum_{j=0}^{m-1} \left( \frac{1}{2^k} \right)^j N\omega_{k+1}(2^j t; u)_p.
\]
For every \( x \in \mathbb{R} \) and \((y, t) \in \Gamma(x)\), we have
\[
(\Delta^k_{2m} u)(y, t) = (T^k_{2m} - I)^m u(y, t) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (T^j_{2m} u)(y, t),
\]

\[
|N(\Delta^k_{2m} u)(y, t)| \leq \sum_{j=0}^{k} \binom{k}{j} |u(y + 2m h_j, t)| \leq \sum_{j=0}^{k} \binom{k}{j} (N u)(x + 2m h_j)
\]

\[
N(\Delta^k_{2m} u)(x) \leq \sum_{j=0}^{k} \binom{k}{j} (N u)(x + 2m h_j).
\]

Therefore, \(||N(\Delta^k_{2m} u)||_p \leq \sum_{j=0}^{k} \binom{k}{j} ||N u||_p\) by the translation invariance property of \(L^p\) norm. Hence, \(N \omega_k(2^{m}t; u) \leq 2^{k} ||N u||_p\). So
\[
N \omega_k(t; u) \leq \left( \frac{1}{2^k} \right)^{m-1} ||N u||_p + \frac{k^{m-1}}{2} \sum_{j=0}^{m-1} \binom{1}{2^k}^j N \omega_{k+1}(2^{k+1}t; u)_{p}.
\]

Let \(2^{j}t \leq s < 2^{j+1}t\). Since \(N \omega_{k+1}(t; u)_p\) is increasing in \(t\), \(\frac{1}{2^{j+1}t} < \frac{1}{t} \leq \frac{1}{2^{j}t}\) and
\[
N \omega_{k+1}(2^{j}t; u)_p \leq N \omega_{k+1}(s; u)_p.
\]

So
\[
\frac{N \omega_{k+1}(2^{j}t; u)_p}{s^{k+1}} < \frac{N \omega_{k+1}(s; u)_p}{s^{k+1}}
\]

Then
\[
\int_{2^{j}t}^{2^{j+1}t} \frac{N \omega_{k+1}(2^{j}t; u)_p}{(2^{j+1}t)^{k+1}} \, ds \leq \int_{2^{j}t}^{2^{j+1}t} \frac{N \omega_{k+1}(s; u)_p}{s^{k+1}} \, ds.
\]

Therefore,
\[
N \omega_k(t; u) \leq \left( \frac{1}{2^k} \right)^{m-1} ||N u||_p + \frac{k^{m-1}}{2} \sum_{j=0}^{m-1} \frac{1}{2^k}^j 2^{k+1}(2^{j}t)^k \int_{2^{j}t}^{2^{j+1}t} \frac{N \omega_{k+1}(s; u)_p}{s^{k+1}} \, ds
\]

\[
= \left( \frac{1}{2^k} \right)^{m-1} ||N u||_p + k 2^k \int_{t}^{2^{m}t} \frac{N \omega_{k+1}(s; u)_p}{s^{k+1}} \, ds.
\]

As \(m \to \infty\), since \(||N u||_p < \infty\), we have \(N \omega_k(t; u)_p \leq k 2^k \int_{t}^{\infty} \frac{N \omega_{k+1}(s; u)_p}{s^{k+1}} \, ds\), which concludes the proof.

Now we come to our final main result on the necessary and sufficient conditions for \(Nu\) to be in \(L^p(\mathbb{R})\).
Theorem 1.9 Let $u : \mathbb{R}_+^2 \to \mathbb{C}$ be harmonic, $p \in [1, +\infty)$ and fix $n \in \mathbb{N}$. For $\varepsilon > 0$, $\mathbb{R}_\varepsilon^2 = \{(x, y) \in \mathbb{R}_+^2 : x \in \mathbb{R}, y > \varepsilon\}$. Then $Nu \in L^p(\mathbb{R})$ if and only if (1) for all $x \in \mathbb{R}$, $\lim_{t \to +\infty} u(x, t) = 0$, (2) $u$ is bounded on each $\mathbb{R}_\varepsilon^2$ with $\varepsilon > 0$ and (3) $N\omega_n(t; u)_p = O(1)$ as $t \to +\infty$ (which means $\sup_{0 < t < +\infty} N\omega_n(t; u)_p < +\infty$).

Remarks: Condition (1) above says that $u$ vanishes at infinity. Condition (3) alone cannot guarantee that $Nu \in L^p(\mathbb{R})$. Consider constant function $u(y, t) = k$ for each $(y, t) \in \mathbb{R}_+^2$. Then $N\omega_n(t; u)_p = 0$ for each $t > 0$, but $Nu \notin L^p(\mathbb{R})$.

Proof. Suppose $Nu \in L^p(\mathbb{R})$. If $x_0 \in \mathbb{R}$ and $t_0 > 0$, then $|u(x_0, t_0)|^p \leq |(Nu)(y)|^p$ for all $|y - x_0| < t_0$ (i.e. $(x_0, t_0) \in \Gamma(y)$). Then

$$\int_{x_0-t_0}^{x_0+t_0} |u(x_0, t_0)|^p \, dy \leq \int_{x_0-t_0}^{x_0+t_0} |(Nu)(y)|^p \, dy \leq ||Nu||_p^p.$$  

Then $2t_0|u(x_0, t_0)|^p \leq ||Nu||_p^p$. Therefore, for all $x \in \mathbb{R}$, $\lim_{t \to +\infty} u(x, t) = 0$. Hence, $u$ is bounded on each $\mathbb{R}_\varepsilon^2$. For $z \in \mathbb{R}, (y, t) \in \Gamma(z)$ and $h \in \mathbb{R}$, we have

$$(\Delta_h u)(y, t) = \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} u(y + jh, t).$$

Now $(N(\Delta_h u))(z) \leq \sum_{j=0}^r \binom{r}{j} (Nu)(z + jh)$ because $|(y + jh) - (z + jh)| = |y - z| < t$ implies $(y + jh, t) \in \Gamma(z + jh)$. By the translation invariance property of $L^p$ norm, $||N(\Delta_h u)||_p \leq \sum_{j=0}^r \binom{r}{j} ||Nu||_p$. So, we have $\sup_{s > 0} N\omega_n(s; u)_p \leq 2^r ||Nu||_p < +\infty$.

Conversely, if (1),(2),(3) hold, then let $P(x) = 1/(\pi(1 + x^2))$ and $P_t(x) = (1/t)P(x/t)$ for all $x \in \mathbb{R}$ and $t > 0$. Then $\int_{\mathbb{R}} P_t(x) \, dx = 1$ for all $t > 0$. Let $(x, y) \in \mathbb{R}_+^2$, $\omega \in \mathbb{R}$ and $t > 0$, then

$$(\Delta_\omega u)(x, y) = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} u(x + j\omega, y).$$

So

$$\int_{\mathbb{R}} (\Delta_\omega u)(x, y) P_t(\omega) \, d\omega = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \int_{\mathbb{R}} u(x + j\omega, y) P_t(\omega) \, d\omega$$

$$= (-1)^r u(x, y) + F_t(x, y),$$

where $F_t(x, y) = \sum_{j=1}^r (-1)^{r-j} \binom{r}{j} \int_{\mathbb{R}} u(x + j\omega, y) P_t(\omega) \, d\omega$.  


Next, we will show for all \((x, y) \in \mathbb{R}^2_+\), \(\lim_{t \to +\infty} F_t(x, y) = 0\). Let \(j = 1, 2, \ldots, r\) and \(N \in \mathbb{N}\) and \((x, y) \in \mathbb{R}^2_+\). By changing \(\xi\) to \(-j\omega\), we get

\[
\int_{\mathbb{R}} u(x + j\omega, y)P_t(\omega)\,d\omega = \int_{\mathbb{R}} u(x - \xi, y)P_t(\xi)\,d\xi = (u(\cdot, y) * P_{tj})(x).
\]

Fix \(\varepsilon_0 > 0\). Consider the functions \(u(x, y + \varepsilon_0)\) and \((u(\cdot, \varepsilon_0) * P_y)(x)\) for all \((x, y) \in \mathbb{R}^2_+\). Since \(u(x, y + \varepsilon_0)\) is a bounded harmonic function on \(\mathbb{R}^2_+\), for some \(f \in L^\infty(\mathbb{R})\), we get \(u(x, y + \varepsilon_0) = \int_{\mathbb{R}} P_y(\omega)f(x - \omega)\,d\omega\). The integral has a non-tangential limit \(f(x)\) a.e. on \(\mathbb{R}\). Since \(u(x, y + \varepsilon_0)\) is continuous on the closure of \(\mathbb{R}^2_+\), \(u(x, \varepsilon_0) = f(x)\) a.e. on \(\mathbb{R}\). Then for all \((x, y) \in \mathbb{R}^2_+, u(x, y + \varepsilon_0) = (u(\cdot, \varepsilon_0) * P_y)(x)\). So for all \((x, y) \in \mathbb{R}^2_+\) and \(j = 1, 2, \ldots, r\),

\[
\lim_{t \to +\infty} \int_{\mathbb{R}} u(x + j\omega, y)P_t(\omega)\,d\omega = \lim_{t \to +\infty} (u(\cdot, y) * P_{tj})(x) = \lim_{t \to +\infty} u(x, y + tj) = 0.
\]

Then for all \((x, y) \in \mathbb{R}^2_+, (*)\) \(\lim_{t \to +\infty} F_t(x, y) = 0\).

Let \(z_0 \in \mathbb{R}, (x, y) \in \Gamma(z_0)\) and \(t > 0\). Since

\[
(-1)^r u(x, y) + F_t(x, y) = \int_{\mathbb{R}} (\Delta^r_{\omega} u)(x, y)P_t(\omega)\,d\omega,
\]

\[
|(-1)^r u(x, y) + F_t(x, y)| \leq \int_{\mathbb{R}} (N(\Delta^r_{\omega} u))(z_0)P_t(\omega)\,d\omega,
\]

so \((N((-1)^r u + F_t))(z_0) \leq \int_{\mathbb{R}} (N(\Delta^r_{\omega} u))P_t(\omega)\,d\omega\). Then by Jensen’s inequality and Fubini’s theorem, we have

\[
\int_{\mathbb{R}} |(N((-1)^r u + F_t))(z)|^p\,dz \leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (N(\Delta^r_{\omega} u))(z)P_t(\omega)\,d\omega \right|^p\,dz
\]

\[
\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |(N(\Delta^r_{\omega} u))(z)|^pP_t(\omega)\,d\omega\,dz = \int_{\mathbb{R}} \|N(\Delta^r_{\omega} u)\|^pP_t(\omega)\,d\omega
\]

\[
\leq \sup_{\omega \in \mathbb{R}} \|N(\Delta^r_{\omega} u)\|^p \leq \left( \sup_{s > 0} N_{\omega_r}(s; u) \right)^p.
\]

Now we claim for all \(z \in \mathbb{R}, |(Nu)(z)|^p \leq \liminf_{n \to +\infty} |(N((-1)^r u + F_{t_n}))(z)|^p\), where \(t_n > 0\) go to +\(\infty\). To see that, let \(z_0 \in \mathbb{R}, \varepsilon_0 > 0\) and \((x, y) \in \Gamma(z_0)\). By (*) \(\lim_{t \to +\infty} F_t(x, y) = 0\). So \(|u(x, y)|^p = \lim_{n \to +\infty} |(-1)^r u(x, y) + F_{t_n}(x, y)|^p\).

Then there is \(k_0 \in \mathbb{N}\) such that for all \(n \geq k_0\), we have \(|u(x, y)|^p - \varepsilon_0 < |(-1)^r u(x, y) + F_{t_n}(x, y)|^p\). Then

\[
|u(x, y)|^p - \varepsilon_0 \leq \inf_{n \geq k_0} |((-1)^r u(x, y) + F_{t_n}(x, y)|^p
\]

\[
\leq \inf_{n \geq k_0} |(N((-1)^r u + F_{t_n}))(z_0)|^p \leq \liminf_{n \to +\infty} |(N((-1)^r u + F_{t_n}))(z_0)|^p.
\]
Since \( z_0 \) and \( \varepsilon_0 \) are independent, \( |(Nu)(z)|^p \leq \liminf_{n \to \infty} |(N((-1)^r u + F_{tn}))(z)|^p \) for all \( z \in \mathbb{R} \). By Fatou’s lemma, we have

\[
\|Nu\|_p^p = \int_{\mathbb{R}} |(Nu)(z)|^p \, dz \leq \int_{\mathbb{R}} \liminf_{n \to \infty} |(N((-1)^r u + F_{tn}))(z)|^p \, dz
\]

\[
\leq \liminf_{n \to \infty} \int_{\mathbb{R}} |(N((-1)^r u + F_{tn}))(z)|^p \, dz \leq \left( \sup_{s > 0} N \omega_r(s; u) \right)_p^p < +\infty.
\]

So \( Nu \in L^p(\mathbb{R}) \).

In the second paper, we will use non-tangential maximal moduli together with the theorem 1.7, 1.8 and 1.9 presented in this paper to characterize those real parts of holomorphic functions defined on upper half space having boundary value functions of higher smoothness order.

**References**


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