Super Fuzzy $n^k$ – Based Graceful Labeling of Some Classes of Trees

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Abstract

Fuzzy sets were introduced by Lotfi. A. Zadeh in 1965, which had a greater improvement in mathematical modeling. In this paper we define the super fuzzy $n^k$– based graceful labeling for path graph, star graph and the product of path and star graph.

Keywords: Star graph, super fuzzy, graceful labeling

1. Introduction

Fuzzy sets were introduced by Lotfi. A. Zadeh in 1965 [1], which had a greater improvement in mathematical modeling. Fuzzy sets and fuzzy relations on it had
been leading to make a fuzzy graph model when there is an ambiguity in vertices
and edges. Narsingh [2] extended the concepts of graph theory. A. Nagoorgani,
Muhammed Akram and D. Rajalakshmi [3, 4] introduced the concepts of labeling
of fuzzy graph. Some results of fuzzy bi-magic and anti-magic labeling on star
graph were proved by K. Thirusangu and D. Jeevitha [5], also fuzzy vertex
graceful labeling are obtained by K. Ameenal and M. Devi on bi star graph [6]. A.
Solairaju and T. Narppasalai [7] introduced super fuzzy 10k - based graceful
labeling for some classes of fuzzy trees. N. Sujatha, C. Dharuman and K.
Thirusangu [8] obtained graceful and magic labeling for special fuzzy graphs.

2. Preliminaries

**Definition 2.1:** Let Y be a space of points with a generic element of Y, y. A fuzzy
set A in Y is characterized by a membership function f_A(y) which associates with
each point in Y a real number in the interval [0,1] with the value of f_A(y) at y,
representing the grade of membership of y in A.

**Definition 2.2:** A graph G = (V,E) is characteristic by a set of vertices V =
{v_1,v_2,...,v_n} and a set of edges E = {e_1,e_2,......,e_m} which coupled a pair
of vertices of V.

**Definition 2.3:** The path graph P_n is a sequence of vertices and edges with no
repetition of its vertices, a path with n vertices has n – 1 as a number of edges.

**Definition 2.4:** The star graph S_n is a tree with n vertices and n – 1 edges, every
vertex has degree one except the center of star which has degree n – 1, also a star
graph is a complete bipartite graph K_1,n.

**Definition 2.5:** Let U and X be two sets, a relation \( \rho \) is said to be a fuzzy relation
from U into X if \( \rho \) is a fuzzy set of U \( \times \) X. A fuzzy graph \( G = (\mu, \sigma) \) is a pair of
functions defined by \( \sigma: V \to [0,1] \) and \( \mu: V \times V \to [0,1], \) for all \( u,v \in V \) we
have \( \mu(u,v) \leq \sigma(u) \wedge \sigma(v). \)

**Definition 2.6:** A graph \( G = (\mu, \sigma) \) is said to be a fuzzy labeling graph if \( \sigma: V \to [0,1] \) and \( \mu: V \times V \to [0,1], \) are bijectives, such that the membership value of
edges and vertices are distinct and \( \mu(u,v) \leq \sigma(u) \wedge \sigma(v) \) for all \( u,v \in V. \)

**Definition 2.7:** A fuzzy labeling graph \( G = (\mu, \sigma) \) is said to be a fuzzy graceful
graph if \( \sigma: V \to [0,1] \) and \( \mu: V \times V \to [0,1], \) such that \( \mu(u,v) = |\sigma(u) - \sigma(v)| \)
and \( \mu(u,v) \) are distinct for all \( u,v \in V. \)

3. Main results

**Theorem 3.1:** Every path \( P_n \) is a super fuzzy \( n^k \) - based graceful.
Proof: Consider a graph $G = P_n$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_iv_{i+1}, i = 1, 2, 3, \ldots, n-1\}$. Assume $w = n, q = n - 1$, where $w$ and $q$ represent number of vertices and edges of the path graph, respectively.

Define $\sigma: V \rightarrow [0, 1]$ by the rules:

(a) $\sigma(v_1) = \frac{w+q}{n} - 1$, where $j = 1, 2, \ldots, n$ (for this case, $j = 1$).

(b) $\sigma(v_2) = \left| \sigma(v_1) - \frac{q^2}{n^2} \right|$, $i, j = 3, 4, \ldots, n$.

Also, define $\mu: V \times V \rightarrow [0, 1]$, for all $u, v \in V$ and $uv$ is an edge in $E$, by:

$\mu(uv) = |\sigma(u) - \sigma(v)|$, then $\sigma, \mu$ both satisfy the supper fuzzy $n^k$-based graceful labeling.

Example 3.1:
The super fuzzy $n^k$-based graceful for $P_3$ is obtained in the following figure. We have $w = 3, q = 2$

![Figure 1: Super fuzzy $n^k$-based graceful labeling for $P_3$.](image)

In this case we have:

$\sigma(v_1) = \frac{5}{3} - 1 = 0.66$

$\sigma(v_2) = \left| 0.66 - \frac{2^2}{3^2} \right| = 0.21$

$\sigma(v_3) = \left| 0.66 - \frac{2^3}{3^3} \right| = 0.36$

And:

$\mu(v_1v_2) = |\sigma(v_1) - \sigma(v_2)| = 0.45$

$\mu(v_2v_3) = |\sigma(v_2) - \sigma(v_3)| = 0.15$

Theorem 3.2: Every star graph $S_n$ is a super fuzzy $n^k$-based graceful labeling.

Proof:
Consider a star graph $S_n$ whose vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2, v_1v_3, \ldots, v_1v_n\}$ with $w = n$ and $q = n - 1$, where $w$ and $q$ are
number of vertices and edges of star graph with index $n$, respectively.

Define a function $\sigma: V \rightarrow [0, 1]$ by

(a) $\sigma(v_i) = \frac{w + q}{n \cdot j} - 1$, where $j = l$

(b) $\sigma(v_i) = \left| \sigma(v_{i-1}) - \frac{q}{n \cdot j} \right|$, $i, j = 2, 3, \ldots, n - 1$

And $\mu: V \times V \rightarrow [0, 1]$, for all $u, v \in V$ and $uv$ is an edge in $E$, by:

$\mu(uv) = |\sigma(u) - \sigma(v)|$, then $\sigma, \mu$ both satisfy the supper fuzzy $n^k$-based graceful labeling.

**Example 3.2**: Consider a star graph $S_n$ with $n = 7$, then $S_7$ is a super fuzzy $n^k$-based graceful labeling. Consider the following star graph with 7 vertices and 6 edges. By applying the set of rules, we have:

$$\sigma(v_1) = \frac{13}{7} - 1 = 0.85$$
$$\sigma(v_2) = \left| 0.85 - \frac{6}{7} \right| = 0.72$$
$$\sigma(v_3) = 0.70, \sigma(v_4) = 0.69, \sigma(v_5) = 0.68, \sigma(v_6) = 0.67$$ and $\sigma(v_7) = 0.66$

$$\mu(v_1v_2) = 0.13, \mu(v_1v_3) = 0.15, \mu(v_1v_4) = 0.16, \mu(v_1v_5) = 0.17, \mu(v_1v_6) = 0.18, \mu(v_1v_7) = 0.19$$

**Theorem 3.3**: The graph $(P_n \ast S_m)$ is a super fuzzy $n^k$-based graceful.

**Proof**: The graph $(P_n \ast S_m)$ has two cases for which $P_n$ is merged with $S_m$. 

![Figure 2: Super fuzzy $n^k$-based graceful labeling for $S_7$](image-url)
Case 1: One of the two vertices of degree one of \( P_n \) is merged with the center vertex of \( S_m \).

Case 2: One of the two vertices of degree one of \( P_n \) is merged with a non-centered vertex of \( S_m \).

In case 1, define \( \sigma: V \to [0, 1] \) and \( \mu: V \times V \to [0, 1] \) by:

\[
\sigma(v_1) = \frac{w + q}{n} - 1
\]

\[
\sigma(v_2) = \left| \sigma(v_1) - \frac{q^2}{n^2} \right|
\]

\[
\sigma(v_i) = \left| \sigma(v_{i-2}) - \frac{q^i}{n^j} \right|, \quad i, j = 3, 4, \ldots, n \text{ with } v_i \text{ a vertex of } P_n.
\]

In second case, define \( \sigma: V \to [0, 1] \) and \( \mu: V \times V \to [0, 1] \) by:

(a) \( \sigma(v_1) = \frac{w + q}{n^j - 1} \), where \( j = 1, 2, \ldots, n \) (for this case, \( j = 1 \)).

(b) \( \sigma(v_2) = \left| \sigma(v_1) - \frac{q^2}{n^2} \right| \)

(b) \( \sigma(v_i) = \left| \sigma(v_{i-2}) - \frac{q^i}{n^j} \right| \)

\[
\mu(v_i v_j) = \left| \sigma(v_i) - \sigma(v_j) \right|, \quad \text{for } v_i v_j \text{ is an edge of } G, \text{ so } \sigma, \mu \text{ both satisfy the super fuzzy } n^k \text{- based graceful labeling for } G = (P_n * S_m).
\]

**Example 3.3:** Consider path \( P_3 \) is merged with \( S_3 \), the result graph has 5 vertices and 4 edges, in either one of the two cases, we have the following label:

![Figure 3](image.png)

Figure 3: Super fuzzy \( n^k \) – based graceful labeling for \((P_3 * S_3)\) of case 1.
Figure 4: Super fuzzy $n^k$ – based graceful labeling for $(P_3 * S_3)$ of case 2.

For case 1, the labeling of vertices and edges are:

$\sigma(v_1) = 0.80, \sigma(v_2) = 0.16, \sigma(v_3) = 0.28, \sigma(v_4) = 0.27$ and $\sigma(v_5) = 0.26$

$\mu(v_1v_2) = 0.64, \mu(v_2v_3) = 0.12, \mu(v_3v_4) = 0.01, \mu(v_3v_5) = 0.02$

For case 2, the labeling of vertices and edges are:

$\sigma(v_1) = 0.80, \sigma(v_2) = 0.16, \sigma(v_3) = 0.28, \sigma(v_4) = 0.24$ and $\sigma(v_5) = 0.04$

$\mu(v_1v_2) = 0.64, \mu(v_2v_3) = 0.12, \mu(v_3v_4) = 0.04, \mu(v_4v_5) = 0.2$

References


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