Pure Mathematical Sciences, Vol. 7, 2018, no. 1, 1 - 9
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/pms.2018.756

Statistically Pre-Cauchy Triple Sequences and Orlicz Functions

Sabiha Tabassum

Department of Applied Mathematics
Zakir Husain College of Engg. and Tech.
A.M.U. Aligarh-202002, India

Abstract

Let $x = (x_{ijk})$ be a triple sequence and M be a bounded Orlicz Function. We prove that x is statistically pre-Cauchy if and only if

$$\lim_{m,n} \frac{1}{m^2n^2t^2} \sum_{i,p\leq m} \sum_{j,q\leq n} \sum_{k,r\leq t} M\left(\frac{|x_{ijk} - x_{pqr}|}{\rho}\right) = 0$$

This implies a theorem due to Connor, Fridy and Kline [13]. The result herein proved are analogous to those by Vakeel A. Khan and Sabiha Tabassum [22].

Keywords: Staistically convergent triple sequence; Statistically pre-Cauchy triple sequence; Orlicz Function

1. Introduction

The concept of statistical convergence was first introduced by Fast[8] and also independently by Buck[17] and Schoenberg[10] for real and complex sequences. Further this concept was studied by Salat[20], Fridy[14], Connor[12,13] and many others.

Statistical convergence is a generalization of the usual notation of convergence that parallels the usual theory of convergence. A sequence $x = (x_i)$ is called statistically convergent to L if

$$\lim_n \frac{1}{n}|i : |x_i - L| \geq \epsilon, i \leq n| = 0$$

and statistically pre-Cauchy if

$$\lim_n \frac{1}{n^2}|(j,i) : |x_i - x_j| \geq \epsilon, j, i \leq n| = 0$$

for every $\epsilon > 0$. Connor, Fridy and Kline[13] proved that statistical convergent sequences are statistically pre-Cauchy and any bounded statistically pre-Cauchy sequence with nowhere dense set of limit points is statistically convergent. They also gave an example showing statistically pre-Cauchy sequences are not necessarily statistically convergent.

A double sequence (x_{kl}) is called statistically convergent to L if

$$\lim_{m,n\to\infty} \frac{1}{mn}|(j,k) : |x_{jk} - L| \geq \epsilon, j \leq m, k \leq n| = 0$$

where the vertical bars indicate the number of elements in the set[6].

Further a double sequence (x_{kl}) is called statistically pre-Cauchy if for every $\epsilon > 0$ there exist $p = p(\epsilon)$ and $q(\epsilon)$ such that

$$\lim_{m,n\to\infty} \frac{1}{m^2n^2}|(j,k) : |x_{jk} - x_{pq}| \geq \epsilon, j \leq m, k \leq n| = 0$$

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$[5,18].

If convexity of M is replaced by $M(x+y) \leq M(x)+M(y)$, then it is called a *Modulus funtion*(see Maddox [9]. An Orlicz function may be bounded or unbounded. For example, $M(x) = x^p(0 < p \leq 1)$ is unbounded and $M(x) = \frac{x}{x+}$ is bounded.

Lindesstrauss and Tzafriri [15] used the idea of Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{ for some } \rho > 0 \right\}$$

which is Banach space with the norm the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

An Orlicz function M satisfies the $\Delta_2 - condition$ ($M \in \Delta_2$ for short) if there exist constant $k \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

2. Results and Discussions

A triple sequence (real or complex) can be defined as a function $X : N \times N \times N \to R(C)$ where N, R and C denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner et. al. [2] and Sahiner and Tripathy [3] and others. Recently Savas and Esi [7] have introduced statistical convergence of triple sequences on probabilistic normed space. Later on, Esi [1] have introduced statistical convergence of triple sequences in topological groups.

Definition 2.1[2] A triple sequence (x_{ijk}) is said to be convergent to L in Pringsheims sense if for every $\epsilon > 0$, there exist $N(\epsilon) \in \mathbb{N}$ such that
$|x_{ijk} - L| < \epsilon$ whenever $i \geq N,\ j \geq N,\ k \geq N$ and write $\lim_{i,j,k\to\infty} x_{ijk} = L$

Remark 2.1 In contrast to the case for single sequences, a P-convergent triple sequences need not be bounded.

Definition 2.2[2] A triple sequence (x_{ijk}) is said to be Cauchy sequence if for every $\epsilon > 0$, there exist $N(\epsilon) \in \mathbb{N}$ such that
$|x_{ijk} - x_{pqr}| < \epsilon$ whenever $i, p \geq N,\ j, q \geq N,\ k, r \geq N$.

Definition 2.3[2] A triple sequence (x_{ijk}) is said to be bounded if there exist $M > 0$, such that $|x_{ijk}| < M$ for all $i, j, k \in \mathbb{N}$.

Definition 2.4[2] A subset E of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have density $\rho(E)$ if the limit given by

$$\rho(E) = \lim_{i,j,k \to \infty} \frac{1}{pqr} \sum_{i \le p} \sum_{j \le q} \sum_{k \le r} \chi_E(i,j,k) \text{exist.}$$

Thus a triple sequence (x_{ijk}) is said to be statistically convergent to L in Pringsheims sense if for every $\epsilon > 0$, $\rho(\{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{ijk} - l| \geq \epsilon\}) = 0$.

Definition 2.5[2] A triple sequence (x_{ijk}) is said to be statistically bounded if there exist $H > 0$ such that

$$\rho(\{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{ijk}| > H\}) = 0.$$

Definition 2.6[2]Thus a triple sequence (x_{ijk}) is said to be statistically Cauchy if for every $\epsilon > 0$ there exist $p = p(\epsilon)$, $q = q(\epsilon)$ and $r = r(\epsilon)$ such that

$$\rho(\{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{ijk} - x_{pqr}| \geq \epsilon\}) = 0.$$

3. Main Results.

A triple sequence (x_{ijk}) is called statistically pre-Cauchy if for every $\epsilon > 0$ there exist $p = p(\epsilon)$, $q(\epsilon)$ and $r(\epsilon)$ such that

$$\lim_{m,n,t \to \infty} \frac{1}{m^2 n^2 t^2} |(i,j,k) : |x_{ijk} - x_{pqr}| \geq \epsilon, i \leq m, j \leq n, t \leq k| = 0.$$

where the vertical bars indicate the number of elements in the set.

In this paper we establish the following criterion for arbitrary triple sequences to be statistically pre-Cauchy.

Theorem 3.1. Let $x = (x_{ijk})$ be a triple sequence and let M be a bounded Orlicz function. Then x is statistically pre-Cauchy if and only if

$$\lim_{m,n,t} \frac{1}{m^2 n^2 t^2} \sum_{i,p \le m} \sum_{j,q \le n} \sum_{k,r \le t} M\left(\frac{|x_{ijk} - x_{pqr}|}{\rho}\right) = 0 \text{ for some } \rho > 0.$$

Proof Suppose that

$$\lim_{m,n,t} \frac{1}{m^2n^2t^2} \sum_{i,p\le m} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right) = 0 \text{ for some } \rho > 0.$$

For each $\epsilon > 0$,$\rho > 0$ and $m, n, t \in \mathbb{N}$, we have
$\lim\limits_{m,n,t} \frac{1}{m^2n^2t^2} \sum_{i,p\le m} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$

$$= \frac{1}{m^2n^2t^2} \sum_{\substack{i,p\le m \\ |x_{ijk}-x_{pqr}|\le\epsilon}} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$$

$$+\frac{1}{m^2n^2t^2} \sum_{\substack{i,p\le m \\ |x_{ijk}-x_{pqr}|\ge\epsilon}} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$$

$$\ge \frac{1}{m^2n^2t^2} \sum_{\substack{i,p\le m \\ |x_{ijk}-x_{pqr}|\ge\epsilon}} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$$

$$\ge M(\epsilon)\left(\frac{1}{m^2n^2t^2}|\{(i,j,k) : |x_{ijk}-x_{pqr}| \ge \epsilon\}, i \le m, j \le n, k \le t\}|\right)$$

$$\ge 0$$

Now suppose that x is statistically pre-Cauchy and that ϵ has been given.
Let $\epsilon > 0$ be such that $M(\delta) < \frac{\epsilon}{2}$.
Since M is bounded, there exist an integer B such that $M(x) < \frac{B}{2}$ for all $x \ge 0$.
Note that, for each $m, n, t \in \mathbb{N}$.
$\frac{1}{m^2n^2t^2} \sum_{i,p\le m} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$

$$= \frac{1}{m^2n^2t^2} \sum_{\substack{i,p\le m \\ |x_{ijk}-x_{pqr}|<\delta}} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$$

$$+\frac{1}{m^2n^2t^2} \sum_{\substack{i,p\le m \\ |x_{ijk}-x_{pqr}|\ge\delta}} \sum_{j,q\le n} \sum_{k,r\le t} M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$$

$$\leq M(\delta)+\frac{1}{m^2n^2t^2}\sum_{i,p\leq m|x_{ijk}-x_{pqr}|\geq\delta}\sum_{j,q\leq n}\sum_{k,r\leq t}M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)$$

$$\leq\frac{\epsilon}{2}+\frac{B}{2}\left(\frac{1}{m^2n^2t^2}|\{(i,j,k):|x_{ijk}-x_{pqr}|\geq\delta,i\leq m,j\leq n,k\leq t\}|\right)$$

$$\leq\epsilon+B\left(\frac{1}{m^2n^2t^2}|\{(i,j,k):|x_{ijk}-x_{pqr}|\geq\delta,i\leq m,j\leq n,k\leq t\}|\right)\quad[3.1]$$

Since x is statistically pre-Cauchy, there is an $\mathbb{N}$ such that R.H.S (3.1) is less than ϵ for all $m,n,t\in\mathbb{N}$.
Hence

$$\lim_{m,n,t}\frac{1}{m^2n^2t^2}\sum_{i,p\leq m}\sum_{j,q\leq n}\sum_{k,r\leq t}M\left(\frac{|x_{ijk}-x_{pqr}|}{\rho}\right)=0$$

Theorem 3.2. Let $x=(x_{ijk})$ be a double sequence and let M be a bounded Orlicz function. Then x is statistically convergent to L if and only if

$$\lim_{m,n,t}\frac{1}{mnn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{t}M\left(\frac{|x_{ijk}-L|}{\rho}\right)=0$$

Proof Suppose that

$$\lim_{m,n,t}\frac{1}{mnn}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{t}M\left(\frac{|x_{ijk}-L|}{\rho}\right)=0$$

with an Orlicz function M,then x is statistically convergent to L (see [12]) Conversely suppose that x is statistically convergent to L. We can prove in the similar manner to Theorem 2.1 that

$$\lim_{m,n,t}\frac{1}{mnt}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{t}M\left(\frac{|x_{ijk}-L|}{\rho}\right)=0$$

using that M is an Orlicz function, we obtain the following corollary.

Corollary 3.3.(see [12,Theorm 3]). Let $x=(x_{ijk})$ be a bounded double sequence. Then x is statistically pre-Cauchy if and only if

$$\lim_{m,n,t} \frac{1}{m^2 n^2 t^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{t} |x_{ijk} - x_{pqr}| = 0$$

Proof. Let $A = \sup_{i,j,k} |x_{ijk}|$ and define

$$M(x) = \frac{(1+2A)x}{1+x}$$

Then

$$M\left(\frac{|x_{ijk} - x_{pqr}|}{\rho}\right) \leq (1+2A)|x_{ijk} - x_{pqr}|$$

and

$$\begin{aligned} M\left(\frac{|x_{ijk} - x_{pqr}|}{\rho}\right) &= (1+2A)\frac{|x_{ijk} - x_{pqr}|}{1+|x_{ijk} - x_{pqr}|} \\ &\geq \frac{\frac{(1+2A)|x_{ijk} - x_{pqr}|}{\rho}}{1+|x_{ijk} - x_{pqr}|} \\ &\geq \frac{(1+2A)|x_{ijk} - x_{pqr}|}{1+2A} \\ &= |x_{ijk} - x_{pqr}| \end{aligned}$$

Hence

$$\lim_{m,n,t} \frac{1}{m^2 n^2 t^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{t} |x_{ijk} - x_{pqr}| = 0$$

if and only if

$$\lim_{m,n,t} \frac{1}{m^2 n^2 t^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{t} M\left(\frac{|x_{ijk} - x_{pqr}|}{\rho}\right) = 0$$

and an immediate application of Theorem (3.1) completes the proof.

Corollary 3.4.(see [11,Theorm 2.1]). Let $x = (x_{ijk})$ be a bounded double sequence. Then x is statistically convergent to L if and only if

$$\lim_{m,n,t} \frac{1}{mnt} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{t} |x_{ijk} - L| = 0$$

Proof. Let $A = \sup_{j,k,k} |x_{ijk}|$ and define

$$M(x) = \frac{(1+A+L)x}{1+x}$$

We can prove in the similar manner as in the proof of Corolary (3.3)

REFERENCES

1. A. Esi, Statistical Convergence of Triple Sequences in Topological Groups, *Annals of the University of Craiova, Mathematics and Computer Science Series,* **40** (2013), no. 1, 29-33.
2. A. Sahiner, M. Gürdal and K. Düden, Triple sequences and their statistical convergence, *Selcuk J. Appl. Math.,* **8** (2007), no. 2, 49-55.
3. A. Sahiner, B.C. Tripathy, Some I-Related Properties of Triple Sequences and their statistical convergence, *Selcuk J. Appl. Math.,* **9** (2008), no. 2, 9-18.
4. A. J. Datta, A. Esi, B. C. Tripathy, Statistical Convergent Triple Sequence Spaces Defined by Orlicz Function, *J. Math. Annal.,* **4** (2013), no. 2, 16-22.
5. B.C. Tripathy and S. Mehta, On a class of sequences related to l^p space defined by Orlicz Functions, *Soochow J. Math.*, **29** (2003), no. 4, 379-391.
6. B.C. Tripathy, Statistically Convergent Double Sequences, *Tamkang J. Math.*, **34** no. 3, (2003), 211-221.
7. E. Savas and A. Esi, Statistical Convergence of Triple Sequences on Probabilistic normed Space, *Annals of the University of Craiova, Mathematics and Computer Science Series,* **39** (2012), no. 2, 226-236.
8. H. Fast, Sur la Convergence Statistiue, *Colloq. Math.*, **2** (1951), 241-244. https://doi.org/10.4064/cm-2-3-4-241-244
9. I.J. Maddox, Sequence Spaces Defined by Modulus, *Math. Proc. Camb. Philosophical Soc.*, **100** (1986), 161-166. https://doi.org/10.1017/s0305004100065968
10. I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66** (1959), 361-375. https://doi.org/10.2307/2308747
11. J.S. Connor, The Statistical and Strong p-Cesaro Convergence of Sequences, *Analysis*, **8** (1988), 47-63. https://doi.org/10.1524/anly.1988.8.12.47

12. J.S. Connor, On Strong Matrix Summability with respect to a Modulus and Statistical Convergence, *Canad. Math. Bull.*, **32** (1989), 194-198. https://doi.org/10.4153/cmb-1989-029-3

13. J.S. Connor, J. Fridy and J. Kline, Statistically Pre-Cauchy Sequences, *Analysis*, **14** (1994), 311-317. https://doi.org/10.1524/anly.1994.14.4.311

14. J.A. Fridy, On Statistical Convergence, *Analysis*, **5** (1985), 301-313. https://doi.org/10.1524/anly.1985.5.4.301

15. J. Lindenstrauss and L. Tzafiri, On Orlicz Sequence Spaces, *Israel J. Math.*, **10** (1971), 379-390. https://doi.org/10.1007/bf02771656

16. M. Basrir and O.Sonalcan, On Some Double Seuence Spaces, *J. Indian Acad. Math.*, **21** (1999), no. 2, 193-200.

17. R.C. Buck, Generalized Asymptote Density, *Amer J. of Math.*, **75** (1953), 335-346. https://doi.org/10.2307/2372456

18. S.D. Parashar and B. Choudhary, Sequence Spaces Defined by Orlicz Functions, *Indian J. Pure Appl. Math.*, **25** (1994), no. 4, 419-428.

19. S. Debnath and B. C. Das, A type of Difference Triple Sequence Spaces, *Palestine Journal of Mathematics*, **4** (2015), no. 2, 284-290.

20. T. Salat, On Statistically Convergent Sequences of Real Numbers, *Math. Soovaco*, **30** (1980), 139-150.

21. Vakeel A. Khan and Q.M. Danish Lohani, Statistically Pre-Cauchy Sequences and Orlicz Functions, *Southeast Asian Bull. of Math.*, **31** (2007), 1107-1112.

22. Vakeel A. Khan and Sabiha Tabassum, Statistically Pre-Cauchy Double Sequences, *Southeast Asian Bull. of Math.*, **36** (2012), 249-254.

Received: May 25, 2017; Published: February 12, 2018