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A Boundary Norm for Weighted Dirichlet Spaces

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Abstract

We give a proof of the equivalence between two norms for the weighted Dirichlet space

In [2], the estimates for the bounds of the weighted Dirichlet Corona Theorem norm is computed using a norm (boundary norm) on the corresponding weighted harmonic Dirichlet space, which is defined on the boundary $\partial\mathbb{D}$ of the unit circle.

In this paper, we stablish the equivalence between the norms of the weighted Dirichlet space and the corresponding harmonic weighted Dirichlet spaces. The approach for the norm equivalence is new and based on 2009 PhD dissertation paper [1]. The main result is *Theorem 2.1*; we use gamma function, Stirling's formulas, uniform convergence of series, Lebesgue dominated convergence theorem and their properties to establish outcomes.

Keywords: Weighted Dirichlet spaces; Boundary norm

1. Preliminaries

Notations: In this paper $\mathbb{D}$ denotes the open unit disc and $\partial\mathbb{D}$ denotes its boundary
Definition: We denote by $\mathcal{D}_\alpha(\mathbb{D})$ or $\mathcal{D}_\alpha$ the weighted Dirichlet space for the weight $\alpha \in (0,1)$ and is defined as:

$$\mathcal{D}_\alpha = \{f \epsilon\, Hol(\mathbb{D}): f(z) = \textstyle\sum_{n=0}^{\infty} f_n z^n \ and\ \sum_{n=0}^{\infty}(n+1)^\alpha |f_n|^2 < \infty\} \quad (1.1)$$

If $f \in \mathcal{D}_\alpha(\mathbb{D})$, then the square of the norm, $\|f\|_{\mathcal{D}_\alpha}^2$ is defined by:

$$\|f\|_{\mathcal{D}_\alpha}^2 = \textstyle\sum_{n=0}^{\infty}(n+1)^\alpha |f_n|^2 \quad (1.2)$$

Similarly, the corresponding weighted **harmonic** Dirichlet space of weight α is denoted by $\mathcal{HD}_\alpha(\mathbb{D})$ or $\mathcal{HD}_\alpha$ and is defined by:

$$\mathcal{HD}_\alpha =$$

$$\left\{f \epsilon\, L^2(\mathbb{T}, d\lambda): f(e^{it}) = \textstyle\sum_{-\infty}^{\infty} \hat{f}(n) e^{int},\ \sum_{n=-\infty}^{\infty}(|n|+1)^\alpha \left|\hat{f}(n)\right|^2 < \infty\right\} \quad (1.3)$$

If $f \in \mathcal{HD}_\alpha(\mathbb{D})$, then the square of the norm, $\|f\|_{\mathcal{HD}_\alpha}^2$ is defined by:

$$\|f\|_{\mathcal{HD}_\alpha}^2 = \textstyle\sum_{n=-\infty}^{\infty}(|n|+1)^\alpha \left|\hat{f}(n)\right|^2 \quad (1.4)$$

Note for $\alpha = 1$, $\mathcal{D}$ is the Dirichlet space, See [4], and

$$\mathcal{D} = \{f \epsilon\, Hol(\mathbb{D}): f(z) = \textstyle\sum_{n=0}^{\infty} f_n z^n \ and\ \sum_{n=0}^{\infty}(n+1)|f_n|^2 < \infty\}$$

If $f \in \mathcal{D}(\mathbb{D})$, then the square of the norm, $\|f\|_{\mathcal{D}}^2$ is defined by,

$$\|f\|_{\mathcal{D}}^2 = \textstyle\sum_{n=0}^{\infty}(n+1)|f_n|^2 \quad (1.5)$$

Is equivalent to

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2\, d\sigma + \int_{\mathcal{D}} |f'(z)|^2\, dA(z), \text{ where } dA(z) = \frac{dm_2(z)}{\pi} \quad (1.6)$$

$$\|f\|_{\mathcal{D}}^2 = \|f\|_{H^2(\mathbb{D})}^2 + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(e^{it}) - f(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^2}\, d\sigma(\theta) d\sigma(t) \quad (1.7)$$

The equivalence between (1.5), (1.6) and (1.7) is a well stablished fact and (1.7) is used to estimate the bounds for the $\mathcal{D}$ corona problem [4].

Definition: Let z be a complex number and $Re(z) > 0$ (the real part of $z > 0$), then the gamma function Γ is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt \quad (1.8)$$

Note the following statements follow from the definition.

- The gamma function converges absolutely
- $\Gamma(z+1) = z\Gamma(z)$

Some facts to consider:

i) If λ is neither a 0 nor a negative integer, then

$$\frac{1}{(1-z)^\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)}{n!\,\Gamma(\lambda)} z^n \quad (1.9)$$

ii) Stirling's Formulas. (1.10)

a) For $|z|$ large enough and when $|arg(z)| < \pi - \varepsilon$,

$$\Gamma(z+1) \approx \sqrt{2\pi z}\,\left(\frac{z}{e}\right)^z$$

b) For large n, $\Gamma(n+1)n! \approx \sqrt{2\pi\, n}\left(\frac{n}{e}\right)^n$

2. Norm Estimates

Let $f \in \mathcal{D}_\alpha(\mathbb{D})$, then, by definition, the square of the norm is

$$\|f\|^2_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty}(n+1)^\alpha |f_n|^2 \tag{2.1}$$

Note: If $f \in H^2(\mathbb{D})$, then $f \in \mathcal{D}_\alpha(\mathbb{D})$. Here $H^2(\mathbb{D})$ is Hardy space.
We claim that (2.1) is equivalently expressed on the boundary, $\partial\mathbb{D}$, of the unit circle by

$$\|f\|^2_{\mathcal{D}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}\, d\sigma(\theta)d\sigma(t) \tag{2.2}$$

Note that

If $f \in \mathcal{D}_\alpha(\mathbb{D})$, then $f(z) = \sum_{n=0}^{\infty} f_n z^n$ for all $z \in \mathbb{D}$ and

If (2.2) holds, then

$$\|f\|^2_{\mathcal{D}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\sum_{n=0}^{\infty} f_n(e^{int} - e^{in\theta})|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}\, d\sigma(\theta)d\sigma(t) \tag{2.3}$$

$$= \|f\|^2_{H^2(\mathbb{D})} + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_n\, \overline{f_k} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(e^{int} - e^{in\theta})(e^{-ikt} - e^{-ik\theta})}{|e^{it}-e^{i\theta}|^{\alpha+1}}\, d\sigma d\sigma \tag{2.4}$$

$$= \|f\|^2_{H^2(\mathbb{D})} + \sum_{n=0}^{\infty}|f_n|^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|e^{int} - e^{in\theta}|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}\, d\sigma(\theta)d\sigma(t) \tag{2.5}$$

In particular, if $f = z^N$, then (2.5) become

$$\|f\|^2_{\mathcal{D}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|e^{iNt} - e^{iN\theta}|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}\, d\sigma(\theta)d\sigma(t) \tag{2.6}$$

By definition $\|z^N\|^2_{\mathcal{D}_\alpha} = (N+1)^\alpha$, then we are done if we show

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|e^{iNt} - e^{iN\theta}|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}\, d\sigma(\theta)d\sigma(t) \sim (N+1)^\alpha \tag{2.7}$$

To this end we show that there are constants C_1 and C_2, depending on α such that

$$C_1(N+1)^\alpha \le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|e^{iNt} - e^{iN\theta}|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}\, d\sigma(\theta)d\sigma(t) \le C_2(N+1)^\alpha \tag{2.8}$$

Let $\beta = (1+\alpha)/2$ and $f(z) = z^N$ and for $0 < r < 1$, define $E(r)$ by

$$E(r) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|1 - r^N e^{iN(\theta - t)}\right|^2}{\left|1 - r\, e^{i(\theta - t)}\right|^{\alpha+1}} d\sigma(\theta) d\sigma(t)$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|1 - r^N e^{iN(\theta - t)}\right|^2}{\left|1 - re^{i(\theta - t)}\right|^{2\beta}} d\sigma(\theta) d\sigma(t) \quad (2.9)$$

By (1.9)

$$E(r) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(\beta+k)\Gamma(\beta+n)}{k!\, n!\, \Gamma(\beta)^2} r^{n+k} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left|1 - r^N e^{iN(\theta - t)}\right|^2 e^{i(n-k)(\theta-t)} d\sigma d\sigma \quad (2.10)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(\beta+k)\Gamma(\beta+n)}{k!\, n!\, \Gamma(\beta)^2} r^{n+k} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \{(1 - r^{2N}) e^{i(n-k)(\theta-t)} - r^N e^{i(N+n-k)(\theta-t)} - r^N e^{-i(N+k-n)(\theta-t)}\} d\sigma d\sigma$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(\beta+k)\Gamma(\beta+n)}{k!\, n!\, \Gamma(\beta)^2} r^{n+k} \left[(1 + r^{2N})\delta_{n,k} - r^N \delta_{k,n+N} - r^N \delta_{n,k+N}\right] \quad (2.11)$$

Where $\delta_{i,j} = \begin{cases} 1, & if\ i = j \\ 0, & otherwise \end{cases}$

Lemma 2.1. Let $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n\, a_k$ be a nonnegative convergent series. Let N be a fixed positive integer and $\{c_{nk}\}$ be a sequence such that, $c_{nk} = \begin{cases} 1, & n = k \\ 1, & n = k + N \\ 1, & k = n + N \\ 0, & otherwise \end{cases}$, then $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(a_n b_k\, c_{nk}) =$

$$\sum_{k=0}^{N}(a_k^2 + 2a_k a_{k+N}) + \sum_{k=N+1}^{\infty}(a_k^2 + 2a_k a_{k+N}) \quad (2.12)$$

Proof: $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n\, a_k = (\sum_{n=0}^{N} a_n + \sum_{n=N+1}^{\infty} a_n)(\sum_{k=0}^{N} a_k + \sum_{k=N+1}^{\infty} a_k)$

$$= [\sum_{n=0}^{N} a_n\ (\sum_{k=0}^{N} a_k + \sum_{k=N+1}^{\infty} a_k)] + [\sum_{n=N+1}^{\infty} a_n\, (\sum_{k=0}^{N} a_k + \sum_{k=N+1}^{\infty} a_k)]$$

$$= \sum_{n=0}^{N} a_n^2 + 2a_0 a_N + 2\sum_{n=1}^{N} a_n a_{n+N} + \sum_{k=N+1}^{\infty} a_k^2 + 2\sum_{K=N+1}^{\infty} a_k a_{k+N}$$ ■

Now, by *Lemma 2.1* $E(r)$ can be expressed as

$$E(r) = \frac{1}{\Gamma(\beta)^2}\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2} r^{2k}(1 + r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!} r^{N+k}\right] + \frac{1}{\Gamma(\beta)^2}\sum_{k=N+1}^{\infty}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2} r^{2k}(1 + r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!} r^{N+k}\right] \quad (2.13)$$

$$= \frac{1}{\Gamma(\beta)^2}\sum_{k=0}^{\infty}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2} r^{2k}(1 + r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!} r^{N+k}\right] \quad (2.14)$$

Let's denote by:

$$S_1(r) = \frac{1}{\Gamma(\beta)^2}\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2} r^{2k}(1+r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!\,(k+N)!} r^{N+k}\right]$$

And

$$S_2(r) = \frac{1}{\Gamma(\beta)^2}\sum_{k=n+1}^{\infty}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2} r^{2k}(1+r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!} r^{N+k}\right]$$

Then

$$E(r) = S_1(r) + S_2(r) \tag{2.15}$$

Lemma 2.2. For $0 < r < 1$,

$$\lim_{r\to 1^-} E(r) = E(1) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|1 - e^{iN(\theta-t)}\right|^2}{\left|1 - e^{i(\theta-t)}\right|^{2\beta}} d\sigma(\theta)d\sigma(t)$$

Proof: Note that $\lim_{r\to 1^-} E(r) = \lim_{r\to 1^-}[S_1(r) + S_2(r)] = \lim_{r\to 1^-} S_1(r) + \lim_{r\to 1^-} S_2(r)$, and that $\lim_{r\to 1^-} S_i(r) = S_i(1)$ for $i = 1, 2$ follows from Lebesgue dominated convergence theorem

Next, we will prove the norm equivalence.

Theorem 2.1. $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|1 - e^{iN(\theta-t)}\right|^2}{\left|1 - e^{i(\theta-t)}\right|^{\alpha+1}} d\sigma(\theta)d\sigma(t) \sim (N+1)^{\alpha}$ (2.16)

Proof: By (2.14)

$$E(r) = \frac{1}{\Gamma(\beta)^2}\sum_{k=0}^{\infty}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2} r^{2k}(1+r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!} r^{N+k}\right]$$

$$= S_1(r) + S_2(r)$$

We will show (2.16) by proving $\lim_{r\to 1^-} S_1(r) = S_1(1) \sim (N+1)^{\alpha}$

Note that for N large enough, since the series (2.14) is absolutely convergent:

$$E(r) = \frac{1}{\Gamma(\beta)^2}\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2} r^{2k}(1+r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!} r^{N+k}\right] + \varepsilon$$

$$= S_1(r) + \varepsilon$$

Where $\varepsilon > 0$ can be made very small.

$$S_1(r) = \sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2}r^{2k}(1+r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!}r^{N+k}\right]$$

$$= \sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)}{k!}r^{k}\left(\frac{\Gamma(\beta+k)}{k!}r^{k}(1+r^{2N}) - 2\frac{\Gamma(\beta+k+N)}{(k+N)!}r^{N}\right)\right]$$

And since,

$$\frac{\Gamma(\beta+k+N)}{(k+N)!} = \frac{\Gamma(\beta+k)}{k!}\times\frac{\beta+N+k-1}{k+N}\times\frac{\beta+N+k-2}{k+N-1}\times\ldots\times\frac{\beta+k}{k+1}$$

$$S_1(r) = \sum_{k=0}^{N}\left[\left(\frac{\Gamma(\beta+k)}{k!}\right)^2 r^k\left(r^k(1+r^{2N}) - 2\frac{\beta+k+N-1}{k+N}\frac{\beta+k+N-2}{k+N-1}\ldots\frac{\beta+k}{k+1}r^N\right)\right]$$

In particular,

$$S_1(1) = 2\sum_{k=0}^{N}\left[\left(\frac{\Gamma(\beta+k)}{k!}\right)^2 - \frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{k!(k+N)!}\right]$$

$$S_1(1) = 2\sum_{k=0}^{N}\left[\left(\frac{\Gamma(\beta+k)}{k!}\right)^2\left(1 - \frac{\beta+N+k-1}{k+N}\times\frac{\beta+N+k-2}{k+N-1}\times\ldots\times\frac{\beta+k}{k+1}\right)\right]$$

Note that,

$$\frac{\beta+N+k-1}{k+N} > \frac{\beta+N+k-2}{k+N-1} > \ldots > \frac{\beta+k}{k+1} \qquad (2.17)$$

And

$$\frac{\beta+k+N-1}{k+N} = 1 - \frac{1-\beta}{k+N}$$

But $\left\{\frac{1-\beta}{k+N}\right\}_{k=1}^{\infty}$ is increasing, which makes $\left\{1-\left(\frac{\beta+k+N-1}{k+N}\right)^N\right\}_{k=1}^{\infty}$ decreasing.

This implies,

$$S_1(1) \geq 2\sum_{k=0}^{N}\left[\left(\frac{\Gamma(\beta+k)}{k!}\right)^2\left(1-\left(\frac{\beta+k+N-1}{k+N}\right)^N\right)\right]$$

$$\geq 2\sum_{k=0}^{N}\left[\left(\frac{\Gamma(\beta+k)}{k!}\right)^2\left(1-\left(1-\frac{1-\beta}{k+N}\right)^N\right)\right]$$

In particular for $k = N$,

$$S_1(1) \geq \sum_{k=0}^{N}\left[\left(\frac{\Gamma(\beta+k)}{k!}\right)^2\left(1-\left(1-\frac{1-\beta}{2N}\right)^N\right)\right]$$

$$\geq \left(1-\left(1-\frac{1-\beta}{2N}\right)^N\right)\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)}{k!}\right]^2 \text{, giving}$$

$$S_1(1) \geq C_0\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)}{k!}\right]^2 \text{ for some constant } C_0 \tag{2.18}$$

On the other hand

$$S_1(1) = 2\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2}\left(1-\frac{\beta+N+k-1}{k+N}\times\frac{\beta+N+k-2}{k+N-1}\times\ldots\times\frac{\beta+k}{k+1}\right)\right]$$

By (2.17)

$$\frac{\beta+N+k-1}{k+N}\times\frac{\beta+N+k-2}{k+N-1}\times\ldots\times\frac{\beta+k}{k+1} \geq \left(\frac{\beta+k}{k+1}\right)^N$$

Which gives,

$$S_1(1) \leq 2\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2}\left(1-\left(\frac{\beta+k}{k+1}\right)^N\right)\right]$$

Furthermore,

$$\left\{1-\left(\frac{\beta+k}{k+1}\right)^N\right\}_{k=1}^{\infty} = \left\{1-\left(\frac{\beta+k+1-1}{k+1}\right)^N\right\}_{k=1}^{\infty} = \left\{1-\left(1-\frac{1-\beta}{k+N}\right)^N\right\}_{k=1}^{\infty}$$

is increasing

Implies that,

$$S_1(1) \leq 2\sum_{k=0}^{N}\left[\left[\frac{\Gamma(\beta+k)}{k!}\right]^2\left(1-\left(\frac{\beta+N}{N+1}\right)^N\right)\right]$$

$$\leq 2\left(1-\left(\frac{\beta+N}{N+1}\right)^N\right)\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)}{k!}\right]^2 \text{ giving,}$$

$$S_1(1) \leq C_1\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2}\right] \text{ for some constant } C_1 \tag{2.19}$$

From (2.18) and (2.19) it follows that

$$C_0\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2}\right] \leq S_1(1) \leq C_1\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2}\right] \tag{2.20}$$

Next, we will show that

$$\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)^2}{(k!)^2}\right] \sim N^{2\beta-1}$$

First note that,

$$\left\{\frac{\Gamma(\beta+k)}{k!}\right\}_{k=1}^{\infty} \text{ is a decreasing sequence}$$

And hence,

$$\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)}{k!}\right]^2 \geq \sum_{k=0}^{N}\left[\frac{\Gamma(\beta+N)}{N!}\right]^2$$

By Stirling formula [3]

$$N\left[\frac{\Gamma(\beta+N)}{N!}\right]^2 \sim NN^{2\beta-2} = N^{2\beta-1}$$

Giving

$$\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)}{k!}\right]^2 \geq \sum_{k=0}^{N}\left[\frac{\Gamma(\beta+N)}{N!}\right]^2 = N\left[\frac{\Gamma(\beta+N)}{N!}\right]^2 \sim NN^{2\beta-2} = N^{2\beta-1} \quad (2.21)$$

On the other hand, Stirling also gives

$$\sum_{k=0}^{N}\left[\frac{\Gamma(\beta+k)}{k!}\right]^2 \sim \sum_{k=0}^{N}\left[k^{\beta-1}\right]^2 \leq NN^{2\beta-2} = N^{2\beta-1} \quad (2.22)$$

From (2.20), (2.21) and (2.22) we get

$$S_1(1) \sim N^{2\beta-1}$$

Which implies that

$$E(1) \sim (N+1)^{2\beta-1}$$

This proves *Theorem 2.1*.

References

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