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On a Transform of Fourier-Stieltjes Type for C^*-Algebra Valued Measures

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Abstract

In this paper we introduce a transformation of Fourier-Stieltjes type acting on C^*-algebra valued measures on a locally compact group. This transformation is related to an action of the group on a Hilbert C^*-module. We obtain among other results an integral representation of a set of bounded operators and the analogue of the convolution theorem.

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1 Introduction

Vector measures play an important rôle in Mathematics. For instance they were revealed crucial in the investigation of the geometric properties of Banach spaces [2]. The Fourier-Stieltjes transform of vector measures on a compact group had been intensely studied in [1]. Fundamental properties of this

transformation were investigated in [1], [7] and [8]. This transformation was introduced via a unitary representation of the group. However, one knows that the category of Hilbert C^*-modules is a natural generalization of the category of Hilbert spaces. Hilbert C^*-modules seem to be also useful in the study of integral transforms (of Fourier type) for C^*-algebra valued measures. Following this idea we introduce here a transformation of Fourier-Stieltjes type in connection with an action of a locally group on a Hilbert C^*-module.
The rest of the paper is organized as follows. The second section recalls some facts about C^*-algebra valued measure and Hilbert C^*-modules while the third one presents the main results consisting in an integral representation of a set of bounded operators and an analogue of the convolution theorem.

2 Preliminaries

In this section we recall some ingredients concerning vector measures and Hilbert C^*-modules that we may need in the sequel.

2.1 C^*-algebra valued measures on locally compact groups

We recall here what is a vector measure. For more details see [2], [3] and [4]. For our purpose we need C^*-algebra valued measures. Let G be a locally compact group. We denote by $\mathfrak{B}(G)$ the σ-field of Borel subsets of G. Let $\mathcal{A}$ be a C^*-algebra. The bracket $\langle x^*, x\rangle$ denotes the duality between $\mathcal{A}$ and its topological dual $\mathcal{A}^*$. A *vector measure* is a countably additive set function $m : \mathfrak{B}(G) \to \mathcal{A}$, that is, for any sequence (A_n) of pairwise disjoint subsets of $\mathfrak{B}(G)$, one has

$$m(\overset{\infty}{\underset{n=1}{\cup}} A_n) = \sum_{n=1}^{\infty} m(A_n), \tag{1}$$

where the convergence holds in the norm topology of $\mathcal{A}$. When m is a vector measure, the measure defined by

$$x^*m(A) := \langle x^*, m(A)\rangle,\ A \in \mathfrak{B}(G), \tag{2}$$

where $x^* \in \mathcal{A}^*$, is a complex measure. A complex Borel-measurable function f defined on G is said to be $m-$integrable if the following conditions are satisfied:

1. $\forall x^* \in \mathcal{A}^*$, f is x^*m-integrable,
2. $\forall A \in \mathfrak{B}(G)$, $\exists\, y \in \mathcal{A}$, $\forall\, x^* \in \mathcal{A}^*$, $\langle x^*, y\rangle = \displaystyle\int_A f dx^*m.$

Then we set $y = \displaystyle\int_G f\, dm.$

An $\mathcal{A}$-valued function f defined on G is said to be m-integrable (Bochner integral) if the function $t \mapsto \|f(t)\|$ is m-integrable.
Let $m : \mathfrak{B}(G) \to \mathcal{A}$ be a vector measure. The *semivariation* [2] of m is the nonnegative function $|m|$ defined on $\mathfrak{B}(G)$ by

$$|m|(A) = \sup\{|x^*m|(A) \, : \, x^* \in \mathcal{A}, \|x^*\| \leq 1\}, \, \forall A \in \mathfrak{B}(G) \tag{3}$$

where $|x^*m|$ is the variation of x^*m, that is

$$|x^*m|(A) = \sup_{\pi} \sum_{\mathfrak{A} \in \pi} |x^*m(\mathfrak{A})| \tag{4}$$

where the supremum is taken over all partitions π of A into finite number of pairwise disjoint members of $\mathfrak{B}(G)$ [2, page 2].
A vector measure m is said to be of *bounded semivariation* if $|m|(G) < \infty$. The range of a vector measure is bounded if and only if it is of bounded semivariation. So a vector measure is said to be bounded if it is of bounded semivariation.

We denote by $M^1(G, \mathcal{A})$ the set of all bounded $\mathcal{A}$-valued measures on G. Let m_i, $i = 1, 2$ be in $M^1(G, \mathcal{A})$. The *convolution product* of m_1 with m_2 is given by :

$$m_1 * m_2(f) = \int \int_G f(st) dm_1(s) dm_2(t) \, , f \in \mathcal{C}(G, \mathcal{A}). \tag{5}$$

where $\mathcal{C}(G, \mathcal{A})$ is the set of $\mathcal{A}-$valued continuous functions on G. the space $M^1(G, \mathcal{A})$ is a complex Banach algebra under the convolution product and the norm

$$\|m\| := |m|(G). \tag{6}$$

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2.2 Hilbert C^*-modules

To generalize the notion of Hilbert space one may consider a space together with a product $\langle \cdot, \cdot \rangle$ which takes values in a C^*-algebra $\mathcal{A}$. This leads to the notion of Hilbert C^*-module.
A pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex vector space E which is also a right $\mathcal{A}$-module such that there is a map

$$E \times E \to \mathcal{A}, \, (X, Y) \mapsto \langle X, Y \rangle$$

with the following properties. For $X, Y, Z \in E$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$,

1. $\langle X, \lambda Y + Z \rangle = \lambda \langle X, Y \rangle + \langle X, Z \rangle$

2. $\langle X, Ya\rangle = \langle X, Y\rangle a$
3. $\langle Y, X\rangle = \langle X, Y\rangle^*$
4. $\langle X, X\rangle$ is a positive element in $\mathcal{A}$
5. $\langle X, X\rangle = 0 \Rightarrow X = 0$.

The equality

$$\|X\| = \|\langle X, X\rangle\|^{\frac{1}{2}} \tag{7}$$

defines a norm on E. If moreover E is complete with respect to this norm then E is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.

Let us give some examples.

1. Every Hilbert space $\mathcal{H}$ is a Hilbert module over $\mathbb{C}$. The inner product is the usual scalar product on $\mathcal{H}$. This shows that Hilbert C^*-module is indeed a generalization of Hilbert space.

2. Every C^*-algebra $\mathcal{A}$ is a Hilbert module over $\mathcal{A}$. The inner product is defined by setting

$$\langle X, Y\rangle = X^*Y, \quad X, Y \in \mathcal{A}. \tag{8}$$

3. Let E_k, $k = 1, \cdots, n$ be Hilbert C^*-modules over the same C^*-algebra $\mathcal{A}$. The direct sum $\overset{n}{\underset{k=1}{\oplus}} E_k$ equipped with the module action and the inner product

$$(x_k) \cdot a = (x_k \cdot a) \text{ and } \langle (x_k), (y_k)\rangle = \sum_{k=1}^{n} \langle x_k, y_k\rangle_{E_k} \tag{9}$$

is a Hilbert C^*-module over $\mathcal{A}$.

See for [5] and [6] for more details about the theory of Hilbert C^*-modules.

3 The ρ-Fourier-Stieltjes transform

This section generalizes some results of [7]. Let G be a locally compact group. We denote by dt a left Haar measure of G. Let $\mathcal{A}$ be a C^*-algebra. Let E be a finite dimensional Hilbert module over $\mathcal{A}$. We fix once and for all an orthonormal basis $(X_1, \ldots, X_n)$ of E as a canonical basis. Let $\rho : G \times E \to, (g, X) \mapsto g \cdot X$ be an action of G on E. We denote the element $g \cdot X$ by $\rho(g)X$. Therefore each $\rho(g)$ is an operator on E. We assume that this operator is adjointable and that $\rho(g)^* = \rho(g^{-1})$ where $\rho(g)^*$ denotes the adjoint of the operator $\rho(g)$. We also assume the following orthogonality relations

$$\int_G \langle \rho(t)^* X_i, X_j \rangle \langle \rho(t) X_k, X_l \rangle dt = \delta_{ik}\delta_{jl},\ i,j,k,l \in \{1,2,\ldots,n\}. \tag{10}$$

where δ_{ij} is the Kronecker symbol.

In the whole paper the integral of $\mathcal{A}$-valued functions is understood as Bochner integral [2]. All the vector measures considered here are $\mathcal{A}$-valued.

The following proposition can be easily proved.

Proposition 3.1 *Let m be a bounded vector measure on G. Then the mapping*

$$(X,Y) \mapsto \int_G \langle \rho(t)^* X, Y \rangle dm(t), \tag{11}$$

is sesquilinear from $E \times E$ into $\mathcal{A}$.

Now one can state the following definition based on an idea from [1].

Definition 3.2 *The ρ-Fourier-Stieltjes transform of a vector measure m is the $\mathcal{A}-$valued sesquilinear map $\mathcal{F}^\rho m$ on $E \times E$ defined by*

$$\mathcal{F}^\rho m(X,Y) = \int_G \langle \rho(t)^* X, Y \rangle dm(t), \quad X,Y \in E. \tag{12}$$

It is possible to tensorize this definition [8]. Let $S(E,E:\mathcal{A})$ be the set of sesquilinear mappings from $E \times E$ into $\mathcal{A}$ and let $\overline{E}$ be the conjugate linear space of E. We endow $E \otimes \overline{E}$ with the projective tensor product norm. Let $\mathcal{B}(E \otimes \overline{E}, \mathcal{A})$ denotes the space of bounded operators from $E \otimes \overline{E}$ into $\mathcal{A}$. Then by the identification $S(E,E:\mathcal{A}) \simeq \mathcal{B}(E \otimes \overline{E}, \mathcal{A})$ one can linearize the transformed measure $\mathcal{F}^\rho m$ by viewing it as the element of the space $\mathcal{B}(E \otimes \overline{E}, \mathcal{A})$ defined by

$$\mathcal{F}^\rho m(X \otimes Y) = \int_G \langle \rho(t)^* X, Y \rangle dm(t), \quad X \in E, Y \in \overline{E} \tag{13}$$

If f is an $\mathcal{A}$-valued integrable function on G then the $\rho-$Fourier transform of f is the $\rho-$Fourier-Stieltjes transform of the vector measure $f dt$, that is

$$\mathcal{F}^\rho f(X \otimes Y) = \int_G \langle \rho(t)^* X, Y \rangle f(t) dt. \tag{14}$$

Definition 3.3 *The $\mathcal{A}$-valued functions $u^\rho_{ij} : t \mapsto \langle \rho(t) X_j, X_i \rangle$ defined on G will be called $\rho-$coefficients.*

The following theorem expresses the $\rho-$Fourier-Stieltjes transform in terms of the $\rho-$coefficients.

Proposition 3.4 *Let m be a bounded vector measure. Then there exists a family $(a_{ij})_{1\leq i,j\leq n}$ of elements of the C^*-algebra $\mathcal{A}$ such that*

$$\mathcal{F}^\rho m = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\mathcal{F}^\rho u_{ij}^\rho. \tag{15}$$

Proof. Let $X \in E$, $Y \in \overline{E}$. We express X and Y in the basis $(X_1, \ldots, X_n)$ that is

$$X = \sum_{k=1}^{n} \beta_k X_k,\ Y = \sum_{l=1}^{n} \gamma_l X_l.$$

Let $m \in M^1(G, \mathcal{A})$. One has

$$\mathcal{F}^\rho m(X \otimes Y) = \sum_{k=1}^{n}\sum_{l=1}^{n} \overline{\beta_k}\gamma_l \mathcal{F}^\rho m(X_k \otimes X_l). \tag{16}$$

In the other hand we have :

$$\begin{aligned}\mathcal{F}^\rho u_{ij}^\rho(X\otimes Y) &= \int_G \langle \rho(t)^* X, Y\rangle\langle\rho(t)X_i, X_j\rangle dt\\ &= \sum_{k=1}^{n}\sum_{l=1}^{n}\overline{\beta_k}\gamma_l\int_G \langle\rho(t)^*X_l, X_k\rangle\langle\rho(t)X_i, X_j\rangle dt\\ &= \overline{\beta_j}\gamma_i\end{aligned}$$

according to the assumption (10). Then we have

$$\mathcal{F}^\rho m(X\otimes Y) = \sum_{j=1}^{n}\sum_{i=1}^{n}\overline{\beta_j}\gamma_i\mathcal{F}^\rho m(X_j \otimes X_i) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\mathcal{F}^\rho u_{ij}^\rho(X\otimes Y)$$

after setting $a_{ij} = \mathcal{F}^\rho m(X_j \otimes X_i)$. $\Box$

Let $\mathfrak{P}^\rho(G, \mathcal{A})$ denotes the space of $\mathcal{A}$-valued functions on G consisting of $\mathcal{A}$-combinaisons of ρ-coefficients, that is the functions of the form

$$f(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{ij}^\rho(\cdot), \tag{17}$$

with $a_{ij} \in \mathcal{A}$. The space $\mathfrak{P}^\rho(G, \mathcal{A})$ is a right $\mathcal{A}$-module. One can endow $\mathfrak{P}^\rho(G, \mathcal{A})$ with the $\mathcal{A}$-valued inner product

$$\langle f, g\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^* b_{ij} \tag{18}$$

where $f(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{ij}^\rho(\cdot)$ and $g(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}u_{ij}^\rho(\cdot)$.

Proposition 3.5 *The space* $(\mathfrak{P}^\rho(G,\mathcal{A}), \langle\cdot,\cdot\rangle)$ *is a pre-Hilbert module over* $\mathcal{A}$.

The proof is straightforward.

Proposition 3.6 *The following equality holds:*

$$\mathcal{F}^\rho(\mathfrak{P}^\rho(G,\mathcal{A})) = \mathcal{B}(E\otimes\overline{E},\mathcal{A}). \tag{19}$$

Proof. Let $T\in\mathcal{B}(E\otimes\overline{E},\mathcal{A})$. Set $a_{ij}=T(X_j\otimes X_i)$ and $S=\sum_{i=1}^{n}\sum_{j=1}^{n}\mathcal{F}^\rho(a_{ij}u^\rho_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\mathcal{F}^\rho u^\rho_{ij}$. Let us show that $T=S$.

$$\begin{aligned} S(X_k\otimes X_l) &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\mathcal{F}^\rho u^\rho_{ij}(X_k\otimes X_l)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\int_G\langle\rho(t)^*X_k,X_l\rangle\langle\rho(t)X_i,X_j\rangle dt\\ &= a_{lk}=T(X_k\otimes X_l).\end{aligned}$$

So $T=S$.
The converse derived from Proposition 3.4. □

From the above proposition we derive an integral representation of elements in $\mathcal{B}(E\otimes\overline{E},\mathcal{A})$.

Corollary 3.7 *The operator* $T\in\mathcal{B}(E\otimes\overline{E},\mathcal{A})$ *if and only if there exists a family* (u^ρ_{ij}) *of* $\rho-$*coefficients and a family* (a_{ij}) *of elements in* $\mathcal{A}$ *such that* $T=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\mathcal{F}^\rho u^\rho_{ij}$,
that is

$$T(X\otimes Y)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\int_G\langle\rho(t)^*X,Y\rangle\langle\rho(t)X_i,X_j\rangle dt. \tag{20}$$

We are going to prove an analogue of the convolution theorem. For each bounded vector measure m, consider the square matrix $(m_{ij})_{1\le i,j\le n}$ with entries in the C^*-algebra $\mathcal{A}$ given by

$$m_{ij}=\mathcal{F}^\rho m(X_j\otimes X_i) \tag{21}$$

Let m^1 and m^2 be two vector measures. We define the product $\times$ as follows : $\mathcal{F}^\rho m^1\times\mathcal{F}^\rho m^2$ is the element of $\mathcal{B}(E\otimes\overline{E},\mathcal{A})$ which is associated with the matrix $(m^1_{ij}).(m^2_{ij})$, the ordinary matrix product of $(m^1_{ij})_{1\le i,j\le n}$ and $(m^2_{ij})_{1\le i,j\le n}$. We can now state the following theorem.

Proposition 3.8 *The following equality holds :*

$$\mathcal{F}^{\rho}(m^1 * m^2) = \mathcal{F}^{\rho}m^2 \times \mathcal{F}^{\rho}m^1. \tag{22}$$

Proof

$$\begin{aligned}
\mathcal{F}^{\rho}(m^1 * m^2)(X_j \otimes X_i) &= \int_G \langle \rho(t)^* X_j, Xi\rangle \, d(m^1 * m^2)(t) \\
&= \int \int_G \langle \rho(st)^* X_j, Xi\rangle dm^1(s) dm^2(t) \\
&= \int \int_G \langle \rho(t)^* \rho(s)^* X_j, Xi\rangle dm^1(s) dm^2(t).
\end{aligned}$$

Now, we express $\rho(s)^* X_j$ in the canonical basis of E:

$$\rho(s)^* X_j = \sum_{k=1}^{n} \langle \rho(s)^* X_j, X_k\rangle X_k. \tag{23}$$

So

$$\begin{aligned}
\mathcal{F}^{\rho}(m^1 * m^2)(X_j \otimes X_i) &= \sum_{k=1}^{n} \int \int_G \langle \rho(s)^* X_j, X_k\rangle \langle \rho(t)^* X_k, X_i\rangle dm^1(s) dm^2(t) \\
&= \sum_{k=1}^{n} \int_G \langle \rho(t)^* X_k, X_i\rangle dm^2(t) \int_G \langle \rho(s)^* X_j, X_k\rangle dm^1(s) \\
&= \sum_{k=1}^{n} \mathcal{F}^{\rho} m^2(X_k \otimes X_i) \mathcal{F}^{\rho} m^1(X_j \otimes X_k).
\end{aligned}$$

Hence $\mathcal{F}^{\rho}(m^1 * m^2) = \mathcal{F}^{\rho}m^2 \times \mathcal{F}^{\rho}m^1$. □

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