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On the Minimal Resolution Conjecture for the Ideal of General Points in $\mathbb{P}^4$

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Abstract

In this paper, we show that the map,

$$\mathrm{H}^0\left(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)\right) \longrightarrow \bigoplus_{i=1}^{s} \Omega^2_{\mathbb{P}^4}(d+2)|_{P_i},$$

is of maximal rank using the method of Horace. This implies that the number of generators of degree $d+2$ of I_S, the ideal of $\{P_1, \cdots, P_s\} \subset \mathbb{P}^4$ is $\left|\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) - 6s\right|$. That is, the betti numbers a_1 and b_1 of the minimal free resolution of I_S satisfy $a_1 b_1 = 0$.

Keywords: maximal rank, minimal free resolution, method of Horace

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1 Introduction

Let $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$, with $s \geq n+1$, be a set of points in general position, and S be the sub-scheme supported at these points. The minimal resolution conjecture asserts that the homogeneous ideal of this sub-scheme, $I_S \subset R = k[x_0, \ldots, x_n]$, where k is an algebraically closed field and R the homogeneous coordinate ring of $\mathbb{P}^n$, has the following expected form;

$$0 \longrightarrow F_{n-1} \cdots \longrightarrow F_p \cdots \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0,$$

where $F_p = R(-d-p)^{a_{p-1}} \bigoplus R(-d-p-1)^{b_p}$, d being the smallest integer satisfying
$s \leq \mathrm{h}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ with s in the d^{th} binomial interval. The non-negative integers a_p and b_p are the graded betti numbers, and they satisfy;

$$a_p = max\left\{0, h^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right) - \mathrm{rk}\left(\Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right)s\right\},$$

and

$$b_p = max\left\{0, \mathrm{rk}\left(\Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right)s - h^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right)\right\}.$$

It has been shown in [11] that the problem of existence of the minimal free resolution of the form above can be reduced to proving that the evaluation map below is of maximal rank for all $0 \leq p \leq n-2$.

$$\mathrm{H}^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)) \longrightarrow \bigoplus_{i=1}^{s} \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)_{|P_i}. \tag{1.1}$$

The minimal resolution conjecture has been proved for $\mathbb{P}^2$ (see [8],[9]), for $\mathbb{P}^3$ (see [1],[2]) and also for $\mathbb{P}^n$ whenever the number of points in consideration s satisfy $n+1 \leq s \leq n+4$, or $s = \binom{n+2}{2} - n$ (see [6] and [10]), or when the number of points s, is sufficiently large compared to the dimension of the projective space, that is, $s >> n$ (see [11]). It is shown in [7] however that the minimal resolution conjecture does not hold in general for $\mathbb{P}^n$, where $n \geq 6$, with $n \neq 9$. For $n = 9$, computational work which has been presented in [4] shows that there are no counterexamples for 50 or fewer points.
The conjecture has also been proved in part for $\mathbb{P}^4$ (see [13] and [14]). In this paper, we present the proof of the remaining part of the minimal resolution conjecture for $\mathbb{P}^4$ for a general set of s points. To be precise, consider the set $\{P_1, P_2, \cdots P_s\}$ of points in general position in $\mathbb{P}^4$ and let $R =$

$k[x_0, x_1, x_2, x_3, x_4]$, where k is an algebraically closed field, be the homogenous coordinate ring of $\mathbb{P}^4$. Then the minimal resolution of the ideal of the sub-scheme of the union of these points has the form;

$$\begin{array}{ccccccccc}
R(-d-3)^{b_2} & & R(-d-2)^{b_1} & & R(-d-1)^{b_0} & & & & \\
\bigoplus & \longrightarrow & \bigoplus & \longrightarrow & \bigoplus & \longrightarrow & I_S & \longrightarrow & 0 \\
R(-d-2)^{a_1} & & R(-d-1)^{a_0} & & R(-d)^{\binom{d+4}{4}-s} & & & & \\
\uparrow & & & & & & & & \\
R(-d-4)^{s-\binom{d+3}{4}} & & & & & & & & \\
\bigoplus & & & & & & & & \\
R(-d-3)^{a_2} & & & & & & & & \\
\uparrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

where the betti numbers a_p and b_p satisfy $a_p b_p = 0$ for $p = 0, 1, 2$. Lauze (1994) and Maingi (2010) have respectively proved that $a_2 b_2 = 0$ and $a_0 b_0 = 0$. In this paper, we prove that $a_1 b_1 = 0$ using the method of Horace. That is, the evaluation map 1.1 is of maximal rank for $p = 1$ and $n = 4$.

This paper is organized as follows. In section 2, we give a brief background of the method of Horace. We then use this method to inductively prove the maximal rank hypothesis for our case in section 3.

2 Preliminaries

The method of Horace was introduced by A. Hirschowitz in a letter to R. Hartshorne in 1984 (which was never published), where he showed maximal rank for the case of 28 points in $\mathbb{P}^3$, that is, the map below is of maximal rank.

$$\mathrm{H}^0\left(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(5)\right) \longrightarrow \bigoplus_{i=1}^{28} \Omega_{\mathbb{P}^3}(5)_{|P_i}.$$

The method has since been used by different authors in different areas. To mention but a few, Maingi [14] used it in proving the minimal resolution conjecture for points in general position in $\mathbb{P}^4$, Ida [12] used it in studying the minimal resolution conjecture for a general set X of points on a smooth quadric in $\mathbb{P}^3$, Ballico and Fontanari [3] used it in error correcting codes among others.

The version of the method of Horace we apply in our work makes use of elementary transformation of vector bundles. We begin by presenting the inductive statements of this method as given in [11].

2.1 General statements for maximal rank hypothesis.

Suppose X is a smooth projective variety and X' is a non-singular divisor of X. Let $\mathcal{F}$ be a locally free sheaf on X and

$$0 \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{F}_{|X'} \longrightarrow \mathcal{F}' \longrightarrow 0$$

be an exact sequence of locally free sheaves on X'. The kernel $\mathcal{E}$ of $\mathcal{F} \longrightarrow \mathcal{F}'$ is a locally free sheaf on X and we have another exact sequence of locally free sheaves on X'

$$0 \longrightarrow \mathcal{F}'(-X') \longrightarrow \mathcal{E}_{|X'} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

and as well exact sequences of coherent sheaves on X

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{F}(-X) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

We then have the following statements.

Hypothesis 2.1.1 ($\mathbf{R}(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}'}, \boldsymbol{y}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$)
Let y, a, b and c be non-negative integers. The statement $\boldsymbol{R}(\mathcal{F}, \mathcal{F}', y; a, b, c)$ asserts that there exists points, $U_1, \cdots, U_a, V_1, \cdots, V_b \in X'$ such that for the quotients

$$\mathcal{F}'_{U_i} \longrightarrow A_i \longrightarrow 0,$$

$$\mathcal{F}_{V_i} \longrightarrow B_i \longrightarrow 0$$

there exists the points $W_1, \cdots, W_c \in X'$ such that for the quotients

$$\mathcal{F}_{W_i} \longrightarrow C_i \longrightarrow 0$$

with the kernel in $ker(F_{W_i} \longrightarrow F'_{W_i})$ then for a non-negative integer z, there exists y points, $Y_1, \cdots, Y_y \in X'$ and z points $Z_1, \cdots, Z_z \in X$ such that the map below is bijective.

$$\mathrm{H}^0(X, \mathcal{F}) \longrightarrow \bigoplus_{i=1}^{a} A_i \oplus \bigoplus_{i=1}^{b} B_i \oplus \bigoplus_{i=1}^{c} C_i \oplus \bigoplus_{i=1}^{y} \mathcal{F}'_{Y_i} \oplus \bigoplus_{i=1}^{z} \mathcal{F}_{Z_i}.$$

Hypothesis 2.1.2 (RD($\boldsymbol{\mathcal{F}, \mathcal{F}', y; a, b, c}$))
Let $y,\ a,\ b$ *and* c *be non-negative integers. The statement* $\boldsymbol{RD}(\mathcal{F}, \mathcal{F}', y; a, b, c)$ *asserts that there exists a points,* $U_1, \cdots, U_a\ V_1, \cdots, V_b \in X'$ *such that for the quotients*

$$\mathcal{F}'_{U_i} \longrightarrow A_i \longrightarrow 0,$$
$$\mathcal{F}_{V_i} \longrightarrow B_i \longrightarrow 0,$$

there exists the points $W_1, \cdots, W_c \in X'$ *such that for the quotients*

$$\gamma(Y) : \mathcal{F}_{W_i} \longrightarrow C_i(Y) \longrightarrow 0$$

with the kernel in $ker(\mathcal{F}_{W_i} \longrightarrow \mathcal{F}'_{W_i})$ *then for a non-negative integer* z*, there exists* y *points,* $Y_1, \cdots, Y_y \in X'$ *and* z *points* $Z_1, \cdots, Z_z \in X$ *such that the map below is bijective.*

$$\mathrm{H}^0(X, \mathcal{F}) \longrightarrow \bigoplus_{i=1}^{a} A_i \oplus \bigoplus_{i=1}^{b} B_i \oplus \bigoplus_{i=1}^{c} C_i(Y_1, \cdots, Y_y) \oplus \bigoplus_{i=1}^{y} \mathcal{F}'_{Y_i} \oplus \bigoplus_{i=1}^{z} \mathcal{F}_{Z_i}.$$

Hypothesis 2.1.3 (RD($\boldsymbol{\mathcal{E}, \mathcal{F}'', y'; a,', b,', c'}$))
Let $y',\ a',\ b'$ *and* c' *be non-negative integers. The statement* $\boldsymbol{RD}(\mathcal{E}, \mathcal{F}'', y'; a', b', c')$ *asserts that there exists* a' *points,* $U_1, \cdots, U_{a'},\ V_1, \cdots, V_{b'} \in X'$ *such that for the quotients*

$$\mathcal{F}''_{U_i} \longrightarrow A_i \longrightarrow 0,$$
$$\mathcal{E}_{V_i} \longrightarrow B_i \longrightarrow 0$$

there exists the points $W_1, \cdots, W'_c \in X'$ *such that for the quotients*

$$\gamma(Y) : \mathcal{E}_{W_i} \longrightarrow C_i(Y) \longrightarrow 0$$

with the kernel in $ker(\mathcal{E}_{W_i} \longrightarrow \mathcal{F}''_{W_i})$ *then for a non-negative integer* z'*, there exists* y' *points,* $Y_1, \cdots, Y'_y \in X'$ *and* z' *points* $Z_1, \cdots, Z'_z \in X$ *such that the map below is bijective.*

$$\mathrm{H}^0(X, \mathcal{F}) \longrightarrow \bigoplus_{i=1}^{a'} A_i \oplus \bigoplus_{i=1}^{b'} B_i \oplus \bigoplus_{i=1}^{c'} C_i(Y_1, \cdots, Y_y) \oplus \bigoplus_{i=1}^{y'} \mathcal{F}''_{Y_i} \oplus \bigoplus_{i=1}^{z'} \mathcal{E}_{Z_i}.$$

These statements together with the theorems below are key in formulating the variants of the method of Horace.

Theorem 2.1.4 (The "simple" method of Horace)
Suppose we have a bijective morphism of vector spaces $\mu : \mathrm{H}^0(X', \mathcal{F}') \longrightarrow L$ *and that we have* $H^1(X, \mathcal{E}) = 0$*. Let* $\mu : \mathrm{H}^0(X, \mathcal{F}) \longrightarrow L$ *be a morphism of vector spaces. Then for*
$H^0(X, \mathcal{F}) \longrightarrow M \oplus L$ *to be of maximal rank it suffices that* $\mathrm{H}^0(X, \mathcal{E}) \longrightarrow M$ *is of maximal rank.*

Theorem 2.1.5 (Differential method of Horace.)
Suppose we are given a surjective morphism of vector spaces,

$$\lambda : H^0\left(X', \mathcal{F}'\right) \longrightarrow L$$

and suppose that there exist a point $Z' \in X'$ *such that*

$$H^0\left(X', \mathcal{F}'\right) \hookrightarrow L \oplus \mathcal{F}'$$

and suppose that $H^1\left(X, \mathcal{E}\right) = 0$. *Then there exist a quotient* $\mathcal{E}(Z') \longrightarrow D(\lambda)$ *with a kernel contained in* $\mathcal{F}'(Z')$ *of dimension* $dim(D(\lambda)) = \mathrm{rk}(\mathcal{F}) - dim(ker\lambda)$ *having the following property. Let* $\mu : H^0\left(X, \mathcal{F}\right) \longrightarrow M$ *be a morphism of vector spaces, then there exist* $Z \in X'$ *such that if* $H^0\left(X, \mathcal{E}\right) \to M \oplus D(\lambda)$ *is of maximal rank then* $H^0\left(X, \mathcal{F}\right) \longrightarrow M \oplus L \oplus \mathcal{F}(Z)$ *is also of maximal rank.*

Remark 2.1.6
In order to apply the method of Horace, one formulates inductive hypotheses analogous to the ones above and proceed to prove them for all possible cases.

2.2 Inductive hypotheses and variants of the method of Horace for $\mathbb{P}^4$

To put the method of Horace into our context let, $X = \mathbb{P}^4$, $X' = \mathbb{P}^3$, $\mathcal{F} = \Omega_{\mathbb{P}^4}(2)$, $\mathcal{F}' = \Omega_{\mathbb{P}^3}(2)$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 4}$, and $\mathcal{F}'' = \mathcal{O}_{\mathbb{P}^3}(1)$. Consider the short exact sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \Omega_{\mathbb{P}^4|\mathbb{P}^3}(2) \longrightarrow \Omega_{\mathbb{P}^3}(2) \longrightarrow 0.$ Taking the wedge product of this sequence by $\Omega_{\mathbb{P}^3}(1)$, we can construct the following diagram of short exact sequences after twisting by $d-1$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^2_{\mathbb{P}^4}(d+1) & = & \Omega^2_{\mathbb{P}^4}(d+1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6} & \longrightarrow & \Omega^2_{\mathbb{P}^4}(d+2) & \longrightarrow & \Omega^2_{\mathbb{P}^3}(d+2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_{\mathbb{P}^3}(d+1) & \longrightarrow & \Omega^2_{\mathbb{P}^4|\mathbb{P}^3}(d+2) & \longrightarrow & \Omega^2_{\mathbb{P}^3}(d+2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We thus have the following inductive hypotheses;

Hypothesis 2.2.1 ($\boldsymbol{H(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2); a, b, c)}$)
The statement $H(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2); a, b, c)$ asserts that there exist $A_1 \cdots A_a \in \mathbb{P}^4$ and $B_1 \cdots B_b \in \mathbb{P}^3$ and a quotient $\Gamma|_C$ of a point in $\mathbb{P}^3$ of dimension θ, (where $1 \leq \theta \leq 5,\ \theta \neq 3$) if $c = 1$ such that the map below is bijective.

$$\mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) \longrightarrow \bigoplus_{i=1}^{a} \Omega^2_{\mathbb{P}^4}(d+2)|_{A_i} \oplus \bigoplus_{j=1}^{b} \Omega^2_{\mathbb{P}^3}(d+2)|_{B_j} \oplus \Gamma|_C.$$

Hypothesis 2.2.2 ($\boldsymbol{H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)}$)
The statement $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$ asserts that there exist $E_1, \cdots, E_e \in \mathbb{P}^4$ and $F_1, \cdots, F_f \in \mathbb{P}^3$ and a quotient $\Gamma|_G$ of a point in $\mathbb{P}^3$ of dimension ϵ, (where $1 \leq \epsilon \leq 5,\ \epsilon \neq 3$) if $g = 1$ such that the map below is bijective.

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{E_i} \oplus \bigoplus_{j=1}^{f} \Omega_{\mathbb{P}^3}(d+1)|_{F_j} \oplus \Gamma|_G.$$

Lemma 2.2.3
If $H(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2); a, b, c)$ is true, then we have the following;

$$3b + \theta c \leq \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)).$$

$$3b + \theta c \equiv \mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) (mod\ 6).$$

$$a = \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) - 3b - \theta c\right) \geq 0$$

where $\theta = 1$ or 4 represents the dimension of the quotient.

Proof. Consider the diagram of the exact sequences below;

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)) & \xrightarrow{\quad\rho\quad} & \bigoplus_{j=1}^{b} \Omega^2_{\mathbb{P}^3}(d+2)|_{B_j} \oplus \Gamma|_C \\
{\scriptstyle \alpha_2}\uparrow & & {\scriptstyle \beta_2}\uparrow \\
H^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) & \xrightarrow{\ \mu\ } & \bigoplus_{i=1}^{a} \Omega^2_{\mathbb{P}^4}(d+2)|_{A_i} \oplus \bigoplus_{j=1}^{b} \Omega^2_{\mathbb{P}^3}(d+2)|_{B_j} \oplus \Gamma|_C \\
{\scriptstyle \alpha_1}\uparrow & & {\scriptstyle \beta_1}\uparrow \\
\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) & \xrightarrow{\quad\tau\quad} & \bigoplus_{i=1}^{a} \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{A_i} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Since α_2 and β_2 are surjective, and by bijectivity of μ, we have that ρ is surjective. Consequently $3b + \theta c \leq \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2))$.
Also by bijectivity of μ, we have that,

$$\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) = 6a + 3b + \theta c. \tag{2.1}$$

Rearranging equation 2.1, we have,

$$\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) - (3b + \theta c) = 6a, \tag{2.2}$$

which is equivalent to saying that $3b + \theta c \equiv h^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1))(mod\ 6)$. Also by multiplying both sides of equation 2.2 by $\frac{1}{6}$, we have $a = \frac{1}{6}(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) - (3b + \theta c))$.
We finally show that $a \geq 0$. Since $a = \frac{1}{6}(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2) - (3b + \theta c))$ and $3b + \theta c \leq \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2))$, we have that,

$$\begin{aligned}
a &= \frac{1}{6}(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) - 3b - \theta c), \\
&\geq \frac{1}{6}(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) - \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2))), \\
&\geq 0.
\end{aligned}$$

Lemma 2.2.4
If $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$ is true, then we have the following;

$$g = 0.$$

$$3f \leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$$

$$e = \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f\right).$$

Proof. The conclusion that $g = 0$ follows from the fact that
$\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \equiv 0(mod\ 6)$.

To show that $3f \leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$, consider the sequences below,

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) & \xrightarrow{\bar{\rho}} & \bigoplus_{j=1}^{f} \Omega_{\mathbb{P}^3}(d+1)|_{F_j} \oplus \Omega_{\mathbb{P}^3}(d+1)|_G \\
\bar{\alpha_2}\uparrow & & \bar{\beta_2}\uparrow \\
\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) & \xrightarrow{\tau} & \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{E_i} \oplus \bigoplus_{j=1}^{f} \Omega_{\mathbb{P}^3}(d+1)|_{F_j} \oplus \Gamma|_G \\
\bar{\alpha_1}\uparrow & & \bar{\beta_1}\uparrow \\
\mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)) & \xrightarrow{\bar{\mu}} & \bigoplus_{i=1}^{e} \Omega^2_{\mathbb{P}^4}(d+1)|_{E_i} \oplus \Gamma'|_G \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

with $\bar{\alpha_2}$ and $\bar{\beta}_2$ surjective, $\bar{\alpha_1}$ and $\bar{\beta}_1$ injective. By bijectivity of τ, $\bar{\rho}$ is surjective, hence
$3f \leq h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$.
Also, since τ is bijective, we have that;

$$\begin{aligned}
\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) &= 6e + 3f, \\
\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f &= 6e, \\
\frac{1}{6}(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f) &= e.
\end{aligned}$$

Finally, using the fact that $e = \frac{1}{6}(\mathbb{P}^4, \mathrm{h}^0(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f)$ and
$3f \leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$, we proceed as follows to show that e is non-negative.

$$\begin{aligned}
e &= \frac{1}{6}(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f), \\
&\geq \frac{1}{6}(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))), \\
&\geq 0.
\end{aligned}$$

since $\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \geq h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$.

3 Main results

In this section, we prove inductively that the evaluation map;

$$\mathrm{H}^0\left(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)\right) \longrightarrow \bigoplus_{i=1}^{s} \Omega_{\mathbb{P}^4}(d+2)|_{P_i},$$

is of maximal rank for a general set $\{P_1, \cdots P_s\}$ of points in $\mathbb{P}^4$, where $s \geq 5$.

3.1 Inductive steps

Lemma 3.1.1
Suppose d, a, b and c satisfy the conditions of lemma 2.2.3.
Write $\mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)) - 3b - \theta c = 3f$, *where f and θ are non negative integers, with $\theta = 1$ or 4. Set $e = a - f$. If $a \geq 0$ and $f \leq \frac{1}{3}h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$, then $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$ implies $H(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2); a, b, c)$.*

Proof. Consider the sequence below;

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6} \longrightarrow \Omega^2_{\mathbb{P}^4}(d+2) \longrightarrow \Omega^2_{\mathbb{P}^3}(d+2) \longrightarrow 0$$

Taking global section, we get the sequence,

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)) \longrightarrow 0$$

Let A be a set of a general points in $\mathbb{P}^4$ and B and C be a general set of points in $\mathbb{P}^3$. Specialize A to $E \cup F$, where E and F are e and f points in general position in $\mathbb{P}^4$ and $\mathbb{P}^3$ respectively. Since $\mathrm{rk}\,\Omega^2_{\mathbb{P}^3}(d+2) \equiv \theta(mod\ 3)$, it is possible to specialize enough points to $\mathbb{P}^3$ to make the map

$$\mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^2}(d+2)|_{B\cup F\cup C})$$

bijective. We can then construct the following diagram;

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)) & \xrightarrow{\quad\gamma\quad} & \mathcal{H} \oplus \mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)|_F) \\
\uparrow & & \uparrow \\
\mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)) & \xrightarrow{\beta} & \mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2)|_A) \oplus \mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)|_B) \oplus \Gamma|_C \\
\uparrow & & \uparrow \\
\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) & \xrightarrow{\alpha} & \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_E) \oplus \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_F) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Where $\mathcal{H} = \mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)|_B) \oplus \Gamma|_C$. The map γ is bijective and if α is bijective, then so is β. Thus $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$ implies $H(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2); a, b, c)$.

By lemma 2.2.4, $3f \le h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$, so that $f \le \frac{1}{3}h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$.
Next, we show that $e \ge 0$. We have by hypothesis that;

$$\begin{aligned} e =& a - f \\ =& \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2) - 3b - \theta c\right) - f \ \textit{since}\ a = \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2) - 3b - \theta c\right), \\ =& \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2) - \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2) + 3f)\right) - f \ \textit{since}\ 3b + \theta c = \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)) - 3f, \\ =& \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+2) - \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2))\right) - \frac{1}{2}f \\ \ge& \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6})\right) - \frac{1}{6}\,\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)), \\ =& \,\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)) - \frac{1}{6}\,\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)), \\ =& \frac{d(d+1)(d+2)(d+3)}{24} - \frac{d(d+1)(d+2)}{12}, \\ =& \frac{d(d+2)(d+3)(d-1)}{24} \ge 0 \ for\ d \ge 1\ hence\ e \ge 0\ for\ d \ge 2. \end{aligned}$$

Lemma 3.1.2
Suppose d, e, f and g satisfy the conditions of lemma 2.2.4. Write $\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - 3f = 3\overline{b} + \epsilon\overline{c}$ where $\overline{b} \ge 0$, $\overline{c} = 1$ or 1, $\epsilon \in \{0, 2, 5\}$ with $\epsilon \in \{2, 5\}$ when $\overline{c} = 1$. Set $\overline{a} = e - \overline{b}$. If $\overline{a} \ge 0$, and $3\overline{b} + (6-\epsilon)\overline{c} \le h^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2))$, then $H(\Omega^2_{\mathbb{P}^4}(d+1), \Omega^2_{\mathbb{P}^3}(d+1); \bar{a}, \bar{b}, \bar{c})$ implies $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega^2_{\mathbb{P}^4}(d+1) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6} \longrightarrow \Omega_{\mathbb{P}^3}(d+1) \longrightarrow 0,$$

Taking global sections, we get the sequence,

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)) \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) \longrightarrow 0,$$

Suppose E is a set of e general points in $\mathbb{P}^4$ and F a general set of f points in $\mathbb{P}^3$. Specialize E to $\bar{A} \cup \bar{B} \cup \bar{C}$, where $\bar{A}$ and $\bar{B}$ are $\bar{a}$ and $\bar{b}$ points in general position in $\mathbb{P}^4$ and $\mathbb{P}^3$ respectively. The map

$$\mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) \longrightarrow L = \mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)|_F)$$

is surjective while the map

$$\mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) \longrightarrow L \oplus \mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)|_{\bar{B}}) \oplus \Gamma|_{\bar{C}}(\bar{B})$$

is injective, where $dim(\Gamma) = 2$ or 5 or 0. Therefore by lemma 2.1.5 there exist a quotient $\Omega^2_{\mathbb{P}^4}(d+1) \longrightarrow \Gamma'|_{\bar{C}}(\bar{B})$ of dimension 4, 1 or 0 such that the map

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_E) \oplus \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_{F_j})$$

is bijective if

$$\mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)) \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)|_{\bar{A}}) \oplus \mathrm{H}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+1)|_{\bar{B}}) \oplus \Gamma'|_{\bar{C}}(B)$$

is bijective. Hence
$H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$ implies $H(\Omega^2_{\mathbb{P}^4}(d+1), \Omega^2_{\mathbb{P}^3}(d+1); \bar{a}, \bar{b}, \bar{c})$.
Note that if $dim(\Gamma) = 0$, then there are no quotients and $3\bar{b} \leq h^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2))$. If $dim(\Gamma) = \epsilon$ where $\epsilon = 2$ or 5, then $dim(\Gamma') = (6-\epsilon)$ so that $3\bar{b} + (6-\epsilon)\bar{c} \leq h^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2))$.
We now show that $\bar{a} = e - \bar{b} \geq 0$. We first note that $e = \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f\right)$ by lemma 2.2.4, and $\bar{b} \leq \frac{1}{3}\,\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - f$ by hypothesis. We thus have that;

$$\begin{aligned}
\bar{a} =& e - \bar{b}, \\
\geq& \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f\right) - \frac{1}{3}\,\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) + f, \\
=& \,\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)) - \frac{1}{3}\,\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) + \frac{1}{2}f, \\
\geq& \,\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)) - \frac{1}{3}\,\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)), \\
=& \frac{d(d+1)(d+2)(d+3)}{24} - \frac{d(d+2)(d+3)}{6} \\
=& \frac{d(d+2)(d+3)(d-3)}{24} \geq 0 \; for \; d \geq 3.
\end{aligned}$$

Remark 3.1.3

Under the conditions of lemma 3.1.2, we have that the lemma fails when;

i. $3\bar{b} = \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - 3f,$

ii. $3\bar{b} + \epsilon\bar{c} = \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - 3f$ *where* $\epsilon \in \{2, 5\}$.

When $3\bar{b} = \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - 3f$, *we also have that* $3\bar{b} \leq \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+1))$. *From these two conditions, we have that;*

$$\begin{aligned}
3f \leq& \,\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+1)), \\
=& \frac{d(d+2)(d+3)}{2} - \frac{(d-1)d(d+2)}{2}, \\
=& 2d(d+2).
\end{aligned}$$

Therefore $f \geq \frac{2}{3}d(d+2)$.

When $3\bar{b}+\epsilon\bar{c} = \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - 3f$, *then either;*

$$\begin{aligned} 3\bar{b}+2\bar{c} &= \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - 3f, \\ 3\bar{b}+4\bar{c} &\leq \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+1)), \end{aligned}$$

or

$$\begin{aligned} 3\bar{b}+5\bar{c} &= \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - 3f, \\ 3\bar{b}+\bar{c} &\leq \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+1)) \end{aligned}$$

For the first case,

$$\begin{aligned} 3f \leq& \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) - \mathrm{h}^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+1)) + 2, \\ =&\frac{d(d+2)(d+3)}{2} - \frac{(d-1)d(d+2)}{2} + 2, \\ =&2d(d+2)+2. \end{aligned}$$

Therefore $f \geq \frac{2}{3}\left(d(d+2)+1\right) = \frac{2}{3}(d+1)^2$.

For the latter case, $f \geq \frac{2}{3}(d+1)^2 - 1$.
Note that in all casses, $f \geq \frac{2}{3}d(d+2)$, *exept when* $d = 5, 8, 11, \cdots$ *for which* $f = \frac{2}{3}(d+1)^2$ *or* $f \geq \frac{2}{3}(d+1)^2 - 1$ *depending on the dimension of the quotient.*

The following lemma will be useful especially when lemma 3.1.2 fails.

Lemma 3.1.4
Suppose d, e, f, g *are non-negative integers satisfying the conditions of lemma 2.2.4. Set* $e' = e - \mathrm{h}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1))$ *if* $e' \geq 0$ *and* $3f \leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d))$, *then* $H(\mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d); e', f, g)$ *implies* $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$

Proof. We choose a hyper-plane $H \subset \mathbb{P}^3$ disjoint from $F_1, F_2, \cdots, F_f$ and specialize m of the e points so that the map $\mathrm{H}^0(H, \mathcal{O}_H) \longrightarrow \oplus_{j=1}^m \mathcal{O}_H|_{E_1}$ is bijective. Twisting $d-1$ by 1, we can construct the following short exact sequence;

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6} \longrightarrow \mathcal{O}_H(d-1)^{\oplus 6} \longrightarrow 0$$

Taking global sections, and evaluating at the corresponding points, we obtain the following diagram;

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \gamma\uparrow \\
\mathrm{H}^0(H, \mathcal{O}_H(d-1)^{\oplus 6}) & \longrightarrow & \bigoplus_{i=1}^{m} \mathcal{O}_H(d-1)^{\oplus 6}|_{E_i} \\
\uparrow & & \beta\uparrow \\
\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) & \longrightarrow & \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{E_i} \oplus \bigoplus_{j=1}^{f} \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_{F_j} \\
\uparrow & & \alpha\uparrow \\
\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6}) & \longrightarrow & \bigoplus_{i=1}^{e'} \mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6}|_{E_i} \oplus \bigoplus_{j=1}^{f} \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_{F_j} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

where $e' = e - m$. From the construction of this diagram, the map γ is bijective, and if the map α is bijective, then so is β which gives $H(\mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d); e', f, g)$ implies
$H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$
With the given hypothesis, if $3f \leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d))$, we show that $e' \geq 0$.

$$
\begin{aligned}
e' =& e - \mathrm{h}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)), \\
=& \frac{1}{6}\left(\mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) - 3f\right) - \mathrm{h}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)), \\
=& \mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)) - \mathrm{h}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)) - \frac{1}{2}f, \\
\geq& \mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)) - \mathrm{h}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)) - \frac{1}{2}\left(\frac{1}{3}\mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d))\right) \\
=& \frac{d(d+1)(d+2)(d+3)}{24} - \frac{d(d+1)(d+2)}{6} - \frac{(d-1)(d+1)(d+2)}{12} \\
=& \frac{(d+1)(d+2)(d^2-3d+2)}{24} \\
=& \frac{(d-1)(d-2)(d+1)(d+2)}{24} \geq 0 \ for\ d \geq 1.
\end{aligned}
$$

For the case when lemma 3.1.2 fails, $f < \frac{2}{3}d(d+2)$ or $f < \frac{2}{3}(d+1)^2$. We show that the condition $3f \leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d))$ in these cases hold when $d \geq 5$. It suffices to show this for $f < \frac{2}{3}d(d+2)$. The latter case can be verified in a similar manner.

$$\begin{aligned} 3f &\leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d)), \\ 2d(d+2) &\leq \frac{1}{2}(d-1)(d+1)(d+2), \\ \frac{1}{2}(d-1)(d+1)(d+2) - 2d(d+2) &\geq 0, \\ \frac{1}{2}(d+2)(d^2-4d-1) &\geq 0 \text{ which holds for } d \geq 5. \end{aligned}$$

Theorem 3.1.5

1. $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1))$ *implies* $H(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2))$ *for all* $d \geq 2$.

2. *For* $d \geq 5$, $H(\Omega^2_{\mathbb{P}^4}(d+1), \Omega^2_{\mathbb{P}^3}(d+1))$ *and* $H(\mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6}), \Omega_{\mathbb{P}^3}(d+1))$ *imply* $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1))$

Proof. The proof of this theorem follows from lemma 3.1.1, lemma 3.1.2 and lemma 3.1.4.

Lemma 3.1.6

Consider $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); s_1, s_2, 0)$, *where* $d \geq 1$, s_1 *and* s_2 *are non-negative integers satisfying* $6s_1+3s_2 = \mathrm{h}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6})$ *and* $3s_2 \leq \mathrm{h}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$. *Suppose that the map* $\mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)) \longrightarrow H^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)|_{S_1})$ *is injective and that the map*
$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{S_1})$ *is surjective with a general* $S_1 \subset \mathbb{P}^4$, *then the hypothesis* $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^2}(d+1); s_1, s_2, 0)$ *is true.*

Proof. Consider the short exact sequence below.

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)) \longrightarrow \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) \longrightarrow 0.$$

Taking global sections, we can construct the diagram below.

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & ker\phi & \longrightarrow & \mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) & & \\
 & & & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)) & \longrightarrow & \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) & \longrightarrow & \mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \phi & & & & \\
 & & \mathrm{H}^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(d+1)|_{S_1}) & \longrightarrow & \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{S_1}) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

The map $Ker\phi \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+1))$ is injective. Let $V \subseteq \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3})$ be the image of this map. By hypothesis $dimV = 3s_2$. It then follows from the diagram below that the map $\alpha : V \longrightarrow \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_{S_2})$ is bijective.

$$\begin{array}{ccc} ker\phi & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_{S_2}) & = & \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_{S_2}) \end{array}$$

Consequently, the map
$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow \bigoplus_{i=1}^{s_1} \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{P_i} \oplus \bigoplus_{j=1}^{s_2} \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 3}|_{P_j}$
is bijective and $H(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); s_1, s_2, 0)$ is true.

3.2 Initial cases.

Theorem 3.2.1

1 $H(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2); a, b, c)$ *is true for all* $d \geq 2$ *and this follows from the following cases;*

- *i* $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4); 5, 5, 0)$,
- *ii* $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4), 6, 3, 0)$,
- *iii* $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 18, 6, 0)$,
- *iv* $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 19, 4, 0)$,
- *v* $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 9, 1_1)$,
- *vi* $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 8, 1_4)$,

2 $H'(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1); e, f, g)$ *is true for all* $d \geq 1$ *and this follows from the following cases;*

- *i* $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 4, 2, 0)$,
- *ii* $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 5, 0, 0)$,
- *iii* $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(5); 13, 4, 0)$,
- *iv* $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(5); 12, 6, 0)$,
- *v* $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(5); 11, 8, 0)$ *and*
- *vi* $H(\mathcal{O}_{\mathbb{P}^4}(3)^{\oplus 6}, \Omega_{\mathbb{P}^3}(7); 28, 14, 0)$.

3 For all $d \geq 2$;

- *i* $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 4, 2, 0)$ *implies* $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4); 6, 3, 0)$,
- *ii* $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 5, 0, 0)$ *implies* $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4), 5, 5, 0)$,
- *iii* $H(\mathcal{O}^{(}_{\mathbb{P}^4}2)\oplus 6, \Omega_{\mathbb{P}^3}(4); 13, 4, 0)$ *implies* $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 17, 8, 0)$,
- *iv* $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(4); 12, 6, 0)$ *implies* $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 18, 6, 0)$,
- *v* $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(4); 11, 8, 0)$ *implies* $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 19, 4, 0)$,
- *vi* $H(\mathcal{O}_{\mathbb{P}^4}(3)^{\oplus 6}, \Omega_{\mathbb{P}^3}(5); 28, 14, 0)$ *implies* $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 9, 1_1)$ *and*
- *vii* $H(\mathcal{O}_{\mathbb{P}^4}(3)^{\oplus 6}, \Omega_{\mathbb{P}^3}(5); 28, 14, 0)$ *implies* $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 8, 1_1)$.

Proof. 1) For $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4); 5, 5, 0)$, we apply lemma 3.1.1 to get $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 5, 0, 0)$ implies $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4); 5, 5, 0)$. For $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 5, 0, 0)$, we apply lemma 3.1.4 with $e = 5,\ f = 0\ g = 0$ and $e' = 4$. This yields $H(\mathcal{O}^{\oplus 6}_{\mathbb{P}^4}, \Omega_{\mathbb{P}^3}(3); 1, 0, 0)$ implies $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 5, 0, 0)$. But the map $\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}^{\oplus 6}_{\mathbb{P}^4}) \longrightarrow \mathcal{O}^{\oplus 6}_{\mathbb{P}^4}|_{P_1}$ is bijective, since it is the evaluation of constants at a point. This shows that $H(\mathcal{O}^{\oplus 6}_{\mathbb{P}^4}, \Omega_{\mathbb{P}^2}(3); 1, 0, 0)$ is true and so is $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 5, 0, 0)$ and $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4); 5, 5, 0)$.

2) For $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4), 6, 3, 0)$, we apply lemma 3.1.1 to get $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 4, 2, 0)$ implies $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4), 6, 3, 0)$. For $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 4, 2, 0)$, we apply lemma 3.1.4, to get $H(\mathcal{O}^{\oplus 6}_{\mathbb{P}^4}, \Omega_{\mathbb{P}^3}(2); 0, 2, 0)$ implies $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 4, 2, 0)$. For $H(\mathcal{O}^{\oplus 6}_{\mathbb{P}^4}, \Omega_{\mathbb{P}^3}(2); 0, 2, 0)$, we proceed as follows. Consider the following diagram;

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}^{\oplus 6}_{\mathbb{P}^4}) & \xrightarrow{\ \pi\ } & \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}^{\oplus 3}_{\mathbb{P}^3}|_{\{P_5, P_6\}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ & & \mathcal{O}^{\oplus 3}_{\mathbb{P}^3}|_{P_5} \oplus \mathcal{O}^{\oplus 3}_{\mathbb{P}^3}|_{P_6} & = & \mathcal{O}^{\oplus 3}_{\mathbb{P}^3}|_{P_5} \oplus \mathcal{O}^{\oplus 3}_{\mathbb{P}^3}|_{P_6} & & \end{array}$$

The maps π and β are bijective and since $\alpha = \beta \circ \pi$, we have that α is also bijective so that $H(\mathcal{O}^{\oplus 6}_{\mathbb{P}^4}, \Omega_{\mathbb{P}^3}(2); 0, 2, 0)$ is true. But $H(\mathcal{O}^{\oplus 6}_{\mathbb{P}^4}, \Omega_{\mathbb{P}^3}(2); 0, 2, 0)$ implies $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 4, 2, 0)$. Therefore $H(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(3); 4, 2, 0)$ is true and so is $H(\Omega^2_{\mathbb{P}^4}(4), \Omega^2_{\mathbb{P}^3}(4), 6, 3, 0)$.

3) For $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 18, 6, 0)$, lemma 3.1.1 yields $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(4); 12, 6, 0)$ implies $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 18, 6, 0))$. But $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(4); 12, 6, 0)$ is true by lemma 3.1.6.

4) For $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 17, 8, 0)$, lemma 3.1.1 yields $H(\mathcal{O}^{(}_{\mathbb{P}^4}2) \oplus 6, \Omega_{\mathbb{P}^3}(4); 13, 4, 0)$ implies $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 17, 8, 0)$. But $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(4); 13, 4, 0)$ is true by lemma 3.1.6.

5) For $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 19, 4, 0)$, lemma 3.1.1 yields $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(4); 11, 8, 0)$ implies $H(\Omega^2_{\mathbb{P}^4}(5), \Omega^2_{\mathbb{P}^3}(5); 19, 4, 0)$. But $H(\mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 6}, \Omega_{\mathbb{P}^3}(4); 11, 8, 0)$ is true by lemma 3.1.6

6) For $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 9, 1_1)$, We apply lemma 3.1.1 to get $H(\mathcal{O}_{\mathbb{P}^3}(3)^{\oplus 3}, \Omega_{\mathbb{P}^2}(4); 28, 14, 0)$ implies $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 9, 1_1)$. But $H(\mathcal{O}_{\mathbb{P}^3}(3)^{\oplus 3}, \Omega_{\mathbb{P}^2}(4); 28, 14, 0)$ is true by lemma 3.1.6.

7) For $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 8, 1_4)$, we apply lemma 3.1.1 to get $H(\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 3}, \Omega_{\mathbb{P}^2}(3); 28, 14, 0)$ implies $H(\Omega^2_{\mathbb{P}^4}(6), \Omega^2_{\mathbb{P}^3}(6); 42, 8, 1_4)$, which is true by lemma 3.1.6.

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