Alexandroff Lattice Spaces

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Abstract

In this paper, we introduce and investigate a new type of Alexandroff space using well known type of posets. In this type, topological properties are reflected in lattice properties. We study connectedness, hyper-connectedness, ultracConnectedness, submaximality, extremally disconnected, and separation axioms. We give characterizations for a set to be pre-open, semi-open and $\alpha$-open.

Mathematics Subject Classification: 54D05, 54F05, 03G10

Keywords: Alexandroff space, lattice, connectedness

1 Introduction

An Alexandroff space [11] (briefly $A$-space) $X$ is a topological space in which an arbitrary intersection of open sets is open. In this space, each element $x$ possesses a smallest neighborhood $V(x)$ which is the intersection of all open sets containing $x$. For every $T_0$ $A$-space $(X, \tau)$, there is a corresponding poset $(X, \leq_\tau)$ in one to one and onto way, where each one of them is completely
determined by the other. If \((X, \tau)\) is a \(T_0\) \(A\)-space, we define the corresponding partial order \(\leq_\tau\), called (Alexandroff) specialization order, by: \(a \leq_\tau b\) iff \(a \in \{b\}\) iff \(b \in V(a)\). On the other hand, if \((X, \leq)\) is a poset, then the collection \(B = \{\uparrow x : x \leq P\}\) forms a base for a \(T_0\) \(A\)-space on \(X\), denoted by \(\tau_{\leq}\). In this case, \(V(x) = \uparrow x\). We consider \((X, \tau_{\leq})\) to be a \(T_0\) \(A\)-space \((X, \tau)\) together with its corresponding poset \((X, \leq)\). A poset \((X, \leq)\) satisfies the ascending chain condition (briefly ACC), if for any increasing sequence \(x_1 \leq x_2 \leq x_3 \leq \cdots\) in \(X\), there exists \(k \in N\) such that \(x_k = x_{k+1} = \cdots\). A \(T_0\) \(A\)-space whose corresponding poset satisfies the ACC is called Artinian \(T_0\) \(A\)-space. \cite{8}. An upper bounded \(T_0\) \(A\)-space is introduced as a generalization of an Artinian \(T_0\)-space. A space is upper bounded \(T_0\) \(A\)-space (briefly UB \(T_0\) \(A\)-space) if every chain of points is bounded above \cite{2}. A space is lower bounded \((LB)\) \cite{2} if every chain of points is bounded below. Given a poset \((X, \leq)\), the set of all maximal elements is denoted by \(M(X)\) (or simply by \(M\)) and the set of all minimal elements is denoted by \(m(X)\) (or simply by \(m\)). Moreover, for each \(x \in X\), we define \(\hat{x}\) to be the set of all maximal elements greater than or equal to \(x\) and \(\check{x}\) the set of all minimal elements less than or equal to \(x\). If \(X\) is a UB \(T_0\) \(A\)-space, then \(M \neq \emptyset\) and \(\hat{x} \neq \emptyset \forall x \in X\). Similarly in an LB \(T_0\) \(A\)-spaces, \(m \neq \emptyset\) and \(\check{x} \neq \emptyset \forall x \in X\). A poset \(P\) is said to be a meet semilattice (resp. join semilattice) if the meet \(x \wedge y = \sup\{x, y\}\) (the join \(x \vee y = \inf\{x, y\}\)) exists in \(P \forall x, y \in P\). Hence \(P\) is lattice iff it is both join and meet semilattice. Any finite meet (resp. join) semilattice with top (resp. bottom) element is a lattice. A subset of a lattice is a sublattice if it is closed under the meet and the join operations. Recall that \(X\) is called nodec \cite{4} if all nowhere dense sets are closed, \(X\) is connected if it is not the union of two disjoint non-empty open sets, and it is hyperconnected if no non-empty open sets are disjoint. Every hyperconnected space is connected. \(X\) is called ultra-connected if no non-empty closed sets are disjoint. \(X\) is submaximal \cite{3} iff each element in the corresponding poset is either maximal or minimal, and \(X\) is extremally disconnected if the closure of every open set is open. A Sierpinski space is a finite topological space with two points, only one of which is closed. So it is a \(T_0\) \(A\)-space. A subset \(A\) of a space \((X, \tau)\) is called a semi-open set\cite{9}(resp. a preopen set\cite{1}, an \(\alpha\)-open set\cite{10}) if \(A \subseteq \overline{A}\) (resp. \(A \subseteq \overline{A}\), \(A \subseteq \overline{A}\)). \(A\) a semi-closed (resp. a preclosed, an \(\alpha\)-closed ) set if \(A^c\) is semi-open (resp. preopen,\(\alpha\)-open ). Thus \(A\) is semi-closed (resp. a preclosed, an \(\alpha\)-closed ) if and only if \(\overline{A} \subseteq A\) (resp. \(\overline{A} \subseteq A\), \(\overline{A} \subseteq A\)). The family of all semi-open (resp. preopen, \(\alpha\)-open ) sets in \(X\) is denoted by \(SO(X)\) (resp. \(PO(X)\), \(\tau_\alpha\)). The pre-closure (resp. semi-closure, \(\alpha\)closure) of \(A\), denoted by \(pCl(A)\) (resp. \(sCl(A)\), \(\alpha\)-\(Cl(A)\)), is the smallest preclosed (resp. semi-closed, \(\alpha\)closed ) set contains \(A\). The pre-interior (resp. semi-interior, \(\alpha\) interior) of \(A\), denoted by \(pInt(A)\) (resp. \(sInt(A)\), \(\alpha\)-\(Int(A)\)), is the largest preopen (resp. semi-open, \(\alpha\)open) set contained in \(A\). In \cite{5}, it
has been shown that a set is $\alpha$-open if and only if it is semi-open, and preopen set.

## 2 Alexandroff Lattice Spaces; Definition and Preliminaries

**Definition 2.1.** Suppose that $(X, \tau_{(\leq)})$ be a $T_0$ $A$-space with corresponding poset $(X, \leq)$. We say that $X$ is Alexandroff join lattice space (briefly, $A_{L_j}$-space) (resp. Alexandroff meet lattice space (briefly, $A_{L_m}$-space)) if its corresponding poset $(X, \leq)$ is join-semilattice (resp. meet-semilattice).

**Theorem 2.2.** Let $(X, \tau_{(\leq)})$ be a $T_0$ $A$-space, $a, b \in X$. Then $a \lor b$ exists in the corresponding poset $(X, \leq)$ if and only if there exists $c \in X$ such that $V(c) = V(a) \cap V(b)$.

*Proof.* Set $c = a \lor b$. If $x \in V(c) = \uparrow c$, then $c \leq x$ and so $x \geq a$ and $x \geq b$. Equivalently, $x \in V(a) \cap V(b)$. If $x \in V(a) \cap V(b)$, then $x \in \{a, b\}^u$, so $x \geq c$. Conversely, $c \geq a$ and $c \geq b$, so $\{a, b\}^u \neq \emptyset$. If $x \in \{a, b\}^u$ this implies that $x \in V(a) \cap V(b) = V(c)$ and hence $c \leq x$. Thus $c = \inf \{a, b\}^u$. \qed

**Theorem 2.3.** Let $(X, \tau_{(\leq)})$ be a $T_0$ $A$-space, $a, b \in X$. Then $a \land b$ exists if and only if there exists $d \in P$ such that $\bar{d} = \pi \cap \bar{b}$.

**Definition 2.4.** Suppose that $(X, \tau_{(\leq)})$ be a $T_0$ $A$-space with a corresponding poset $(X, \leq)$. We say that $X$ is Alexandroff lattice space (briefly, $A_L$-space) if the corresponding poset $(X, \leq)$ is lattice. In this case, $X$ is both $A_{L_j}$-space and $A_{L_m}$-space.

Using Theorem 2.2 and Theorem 2.3, we directly get the following theorem.

**Theorem 2.5.** Let $(X, \tau_{(\leq)})$ be an $A$-space. Then $X$ is $A_L$-space iff $\forall a, b \in X$, there exist $c, d \in X$ such that $V(c) = V(a) \cap V(b)$ and $\bar{d} = \pi \cap \bar{b}$.

**Remark 2.6.** If $(P, \leq)$ is a lattice and $A \subseteq P$, then $A$ need not be a lattice under the induced order. So a subspace of $A_L$-space need not be $A_L$-space.

**Example 2.7.** Let $X = \{\top, \bot, a, b, c, d\}$ with order as shown in Figure 1 (a). Then $(X, \leq)$ is a lattice. So the induced $A$-space is $A_L$-space. Let $S = \{a, b, c, d\}$. Then the induced order on $S$ is shown in Figure 1 (b). So $S$ as a poset is not lattice and hence $S$ is not $A_L$-space.
3 Propertis of $A_L$-Spaces

Theorem 3.1. Let $(X, \tau_{(\leq)})$ be an $A_{L_j}$-space. Then $(X, \tau_{(\leq)})$ is hyperconnected space.

Proof. For non-empty open sets $U$ and $V$, let $x \in U$ and $y \in V$. Then $x \lor y = c$ exists in $X$. So $c \in V(x) \subseteq U$ and $c \in V(y) \subseteq V$. Therefore, $U \cap V \neq \emptyset$. □

Corollary 3.2. An $A_{L_j}$-space is always connected space.

Proposition 3.3. Let $(X, \tau_{(\leq)})$ be an $A_{L_m}$-space. Then $(X, \tau_{(\leq)})$ is ultraconnected space.

Proof. Suppose that $U$ and $V$ are two non-empty disjoint closed sets with $x \in U$ and $y \in V$. Then $x \land y \in \{x\} \cap \{y\}$. Hence, $U \cap V \neq \emptyset$. □

Corollary 3.4. Let $(X, \tau_{(\leq)})$ be an $A_L$-space. Then $(X, \tau_{(\leq)})$ is both ultraconnected and hyperconnected space.

The following example gives an ultraconnected and hyperconnected $A$-space which is not $A_L$-space.

Example 3.5. Let $X = \{a, b, c, d, e, f\}$ with order as shown in Figure 2. Then $(X, \tau_{(\leq)})$ is both ultraconnected and hyperconnected $A$-space. But $(X, \tau_{(\leq)})$ is not $A_L$-space, since $b \lor c$ does not exist.

Theorem 3.6. If $X$ is an $A_{L_j}$-space (resp. an $A_{L_m}$-space), then $X$ contains at most one maximal (resp. one minimal) element.

Proof. By Theorem 3.1 $X$ is hyperconnected. If there exist $x \neq y$ in $M$, then both $\{x\}$ and $\{y\}$ are disjoint open sets in $X$, a contradictions. On the other hand, if $X$ is an $A_{L_m}$-space, then $X$ is ultraconnected. So if there exist $x \neq y$ in $m$, then both $\{x\}$ and $\{y\}$ are disjoint closed sets in $X$ which is a contradiction.
Corollary 3.7. An $A_L$-space (resp. $A_{Lm}$-space) is a UB (resp. an LB) iff it has a top element (resp. bottom element).

Theorem 3.8. [7] A UB $A$-space $X$ is nodec iff each element is either maximal or minimal iff $X$ is submaximal.

Theorem 3.9. If an $A_L$-space $X$ is a UB, then the space $(X,\tau(\leq))$ is nodec if and only if $\forall x \neq y \in X, V(x) \cap V(y) = \{\top\}$.

Proof. Suppose that $V(x) \cap V(y) \neq \{\top\}$ for some $x, y$ in $X$. If $z \neq \top$ and $z \in V(x) \cap V(y)$, then $x, y \leq z$. Since $\{z\}$ is closed and $x, y \in \{z\}=\{z\}$, we get $x = y = z$. Conversely, suppose that $z \in \{x\}$ where $z \neq x$, then $z \leq x$. So $V(z) \cap V(x) = V(x) = \{\top\}$. Hence $x = \top$. \qed

Proposition 3.10. If $X$ is UB $T_0$ $A$-space, then $X$ is submaximal iff each element of $X$ is either maximal or minimal in the corresponding post $(X,\leq)$.

Proposition 3.11. Let $X$ be an $A_L$-space with more than one element. Then, $X$ is submaximal iff $X$ is the Sierpinski space.

Proof. A submaximal $T_0$ $A$-space is both UB and LB $A$-space. So, by Theorem 3.6 there exist one maximal element and one minimal element. Conversely, the Sierpinski space is submaximal $A_L$-space. \qed

Corollary 3.12. Let $(X,\tau(\leq))$ be an $A_L$-space. Then the following are equivalent:

1. Each element of $X$ is either maximal or minimal.
2. $X$ is submaximal.
3. $X$ is the Sierpinski space.

Proof. $A_L$-space is UB-space, we get the result from Theorem 3.10 and Proposition 3.11. \qed
Example 3.13. If \((X, \tau(\leq))\) is a countable submaximal \(A_{L_j}\)-space (resp. a countable submaximal \(A_{L_m}\)-space), then by Theorem 3.6 \(|M| = 1\) (resp. \(|m| = 1\)). So the only possible Hasse diagram of \(X\) has a general form as shown in Figure 3 (a) (resp. (b)) below:

![Hasse Diagram](image)

Figure 3: Countable submaximal \(A_{L_j}\) and \(A_{L_m}\) spaces

Remark 3.14. If \(X\) is an \(A_L\)-space with more than one element, then \(X\) is neither discrete space nor regular space.

Proposition 3.15. An \(A_L\)-space is always normal space.

Proof. There is no disjoint closed sets.

We call a function \(f\) between two lattices is a lattice homomorphism if 
\[ f(x \land y) = f(x) \land f(y) \quad \text{and} \quad f(x \lor y) = f(x) \lor f(y) \]
for all \(x, y \in L\). That is, \(f\) preserves \(\land\) and \(\lor\). Clearly each lattice homomorphism is order preserving. To see this, let \(x \leq y\). Then \(x \land y = x\), hence \(f(x) = f(x \land y) = f(x) \land f(y)\), which implies that \(f(x) \leq f(y)\).

Theorem 3.16. [6] Let \(X\) and \(Y\) be \(T_0\) \(A\)-spaces. Then a function \(f : X \to Y\) is continuous at a point \(a\) if and only if \(f(\uparrow a) \subseteq \uparrow f(a)\); that is, for all \(x \in X\), if \(x \geq a\) then \(f(x) \geq f(a)\).

In \(T_0\) \(A\)-spaces, the concept of continuity in the category of topology is equivalent to the concept of order preserving in the category of posets.

Corollary 3.17. Let \(X\) and \(Y\) be two \(A_{L_j}\)-spaces and \(f : X \to Y\). Then \(f\) is continuous iff \(f\) preserves \(\lor\).

Proof. Let \(x \leq y\), then \(x \lor y = y\). Then \(f(x) \leq f(y) = f(x \lor y)\). Hence \(f(x \lor y) = f(y) = f(x) \lor f(y)\). Conversely, if \(f(x \lor y) = f(x) \lor f(y)\) and \(x \leq y\), then \(f(y) = f(x) \lor f(y)\). Hence \(f(x) \leq f(y)\) and \(f\) is continuous.

Corollary 3.18. Let \(X\) and \(Y\) be two \(A_L\)-spaces and \(f : X \to Y\). Then \(f\) is continuous iff \(f\) is lattice homomorphism between the corresponding posets.
4  Boundedness of Alexandroff Lattice Spaces

**Proposition 4.1.** [7] Let \((X, \tau(\leq))\) be a UB \(T_0\) Aspace. Then \(A\) is preopen if and only if \(\hat{x} \subseteq A\ \forall x \in A\).

**Proposition 4.2.** [7] Let \((X, \tau(\leq))\) be a UB \(T_0\) Aspace. Then \(A\) is semi-open set if and only if \(\hat{x} \cap A \neq \emptyset\ \forall x \in A\).

**Theorem 4.3.** [7] A set \(D\) is dense in a UB \(T_0\) Aspace if and only if \(M \subseteq D\).

If \((X, \leq)\) is a lattice, then \(X\) need not has a top element or a bottom element. If \(X\) has a top element, then \(X\) is said to be bounded above, and if \(X\) has a bottom element, then \(X\) is said to be bounded below. \(X\) is called bounded if it is both bounded above and bounded below. For this end, we get the following definition.

**Definition 4.4.** Let \((X, \tau)\) be a \(T_0\) A-space. Then \(X\) is called bounded above (resp. below) \(A_L\)-space if its corresponding poset \((X, \leq)\) is bounded above (resp. below) lattice. \(X\) is bounded \(A_L\)-space (briefly, \(A_{BL}\)-space) if the corresponding poset is bounded lattice.

**Lemma 4.5.** For a subset \(A\) of a \(T_0\) A-space \(X\),

(i) \(A\) has a maximum element iff there exists an element \(b \in A\) such that \(A \subseteq \{b\}\). In this case \(\{b\}\) is dense in \(A\).

(ii) \(A\) has a minimum element iff there exists an element \(a \in A\) such that \(A \subseteq V(a)\).

**Remark 4.6.** Let \((X, \tau_{(\leq)})\) be an A-space, then we say \((X, \tau_{(\leq)})\) is bounded above if there exist \(\top \in X\) such that \(V(\top) \subseteq V(x)\).

**Theorem 4.7.** Let \(X\) be an \(A_L\)-space. Then \(X\) is both UB and LB iff \(X\) is \(A_{BL}\)-space. In this case, \(\hat{x} = \top\) and \(\check{x} = \bot\ \forall x \in X\).

**Proof.** By Corollary 3.7 \(X\) has a top and bottom element. Conversely, if \(X\) is \(A_{BL}\)-space then the corresponding poset has both top element \(\top\) and bottom element \(\bot\). Thus \(X\) is both UB and LB.

**Theorem 4.8.** If \((X, \tau(\leq))\) is a finite \(A_L\) -space then \((X, \tau(\leq))\) is \(A_{BL}\) -space.

**Theorem 4.9.** Let \((X, \tau_{(\leq)})\) be an \(A_L\)-space. Then \(X\) is UB \(A\)-space iff it \(X\) is bounded above. In this case, \(X\) has a dense singleton set.

**Proof.** The corresponding poset \((X, \leq)\) is join semilattice, and the set \(|M| \neq \emptyset\). So by Theorem 3.6, \(|M| = 1\). Conversely, it is true in any \(T_0\) A-space.
Theorem 4.10. Let \((X, \tau_{\leq})\) be a bounded above \(A_{L_j}\)-space and \(A\) a non-empty subset of \(X\). Then:

1. \(A\) is preopen if and only if \(\{\top\} \subseteq A\).

2. \(A\) is semi-open if and only if \(\{\top\} \subseteq A\).

Proof. 1. For all \(x \in A\), \(\hat{x} = \{\top\}\), then by Proposition 4.1 we done.

2. For all \(x \in A\), \(\hat{x} = \{\top\}\) and \(\hat{x} \cap A \neq \emptyset \ \forall x \in A\) if and only if \(\hat{x} = \{\top\} \subseteq A\), then by Proposition 4.2 we done. \(\square\)

Corollary 4.11. Let \(X\) be a UB \(A_{L_j}\)-spaces. Then we have the following:

1. \(PO(X) = SO(X)\); that is, a subset \(A\) of \(X\) is semi-open iff it is preopen.

2. \(X\) is extremally disconnected.

Proof. The proof of the two points comes from the fact that the corresponding poset has a top element \(\top\), so \(\forall x \in X\), \(\hat{x} = \{\top\}\) and so \(|\hat{x}| = 1\). \(\square\)

Corollary 4.12. Let \((X, \tau_{\leq})\) be a bounded above \(A_{L_j}\)-space. Then the set \(A\) is preclosed (resp. semi-closed) if and only if \(A = X\) or \(\top \notin A\).

In general \(PO(X) \neq \tau_{\alpha}\). But they are equal in a UB \(T_0\) \(A\)-space \([7]\).

Corollary 4.13. Let \(X\) be a UB \(A_{L_j}\)-space and \(A\) a nonempty subset of \(X\). Then the following are equivalent:

1. \(A\) is preopen.

2. \(A\) is semi-open.

3. \(\{\top\} \subseteq A\).

4. \(A\) is dense.

Proof. Direct from Corollary 4.11, Theorem 4.10 and Theorem 4.3. \(\square\)

Theorem 4.14. \([7]\) A set \(D\) is dense in a UB \(T_0\) \(A\)-space if and only if \(M \subseteq D\).

Theorem 4.15. Let \((X, \tau_{\leq})\) be an \(A_{L_j}\)-space with top element \(\top\) and \(A\) is a subset of \(X\). Then:

1. \(p\text{Int}(A) = s\text{Int}(A) = \left\{ \begin{array}{ll} A, & \top \in A; \\ \emptyset, & \top \notin A. \end{array} \right.\)

2. \(p\text{Cl}(A) = s\text{Cl}(A) = \left\{ \begin{array}{ll} X, & \top \in A; \\ A, & \top \notin A. \end{array} \right.\)
Proof. (1) Let \( A \subseteq X \). If \( A \) has a top element, then \( A \) is preopen and so \( A = p\text{Int}(A) \). Then for all \( x \in A \) \( \hat{x} = \top \notin A \). If \( p\text{Int}(A) \neq \emptyset \), then \( \top \notin p\text{Int}(A) \subseteq A \) a contradiction, so \( p\text{Int}(A) = \emptyset \).

(2) Let \( A \subseteq X \). such that \( \top \notin A \). By Corollary 4.12, \( A \) is preclosed so \( p\text{Cl}(A) = A \). If \( \top \notin A \) and \( p\text{Cl}(A) \neq X \), then by Corollary 4.12 \( \top \notin p\text{Cl}(A) \). Now \( \top \in A \subseteq p\text{Cl}(A) \) a contradiction, so \( p\text{Cl}(X) = X \).

\[ \square \]

**Theorem 4.16.** Let \((X, \tau_{\leq})\) be a bounded above \(A_{Lj}\)-space. Then we have the following:

1. \( A^o = \emptyset \), iff \( \top \notin A \).
2. The subset \( A \) is nowhere dense if and only if \( \top \notin A \).
3. Any subset is either dense or nowhere dense.

**Theorem 4.17.** [12] Let \((X, \tau_{\leq})\) be a \(T_0\) \(A\)-space. Then \( X \) is compact iff it is \(LB\) and \( m \) is finite.

**Corollary 4.18.** An \( A_{Lm} \)-space is compact iff it is \( LB \).

*Proof.* Direct from Theorem 3.7. Conversely, If \( X \) is \( LB \), then by Corollary 3.7, \( m \) is finite and so \( X \) is compact. \[ \square \]

**Remark 4.19.** A subspace of a bounded \(A\)-space is need not be bounded. That is, a boundedness property is not hereditary property on \(A\)-spaces. As the following example shows.

**Example 4.20.** Let \( X \) be the set of natural numbers with usual order together with the element \( \top \) as top element. Then \( X \) is bounded \(A\)-space. The set of even numbers as a subspace is not bounded \(A\)-space.

**References**


Received: November 23, 2016; Published: January 3, 2017