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Functions of Bounded Radius Rotation Involving Multiplier Transformations

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Abstract

In this article, we use multiplier transformations to define some new classes of analytic functions which map the open unit disk to the domain related with the functions of bounded radius rotation of complex order. Some interesting properties of these classes including the inclusion relations, integral preserving properties, inverse inclusion and their relationships with the previously known results are studied.

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1 Introduction

Let $\mathcal{A}$ represent the class of all analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

defined in the open unit disc $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For functions $f,\ g \in \mathcal{A}$, the convolution $f * g$ is defined by $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. Let $\mathcal{P}$ denote the class of Carathéodory functions p such that $p(0) = 1, \Re(p(z)) > 0$ and $p(z) = 1 + c_1 z + c_2 z^2 + \cdots, z \in \mathbb{U}$. The Möbius function

$$p_0(z) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}), \tag{2}$$

or its rotation acts as an extremal function for the class $\mathcal{P}$ and p_0 maps the open unit disc to the right half-plane. A function $p \in \mathcal{P}_k(b)$, if

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \in \mathcal{P}_k \qquad (k \geq 2, z \in \mathbb{U}). \tag{3}$$

For the details of the class $\mathcal{P}_k$, we refer, [12] and others.[5, 11]. Let $F_\varsigma : \mathcal{A} \to \mathcal{A}$ be defined by

$$F_\varsigma(z) = F_\varsigma(f)(z) = \frac{\varsigma + 1}{z^\varsigma} \int_0^z t^{\varsigma - 1} f(t)\, dt \qquad (\varsigma > -1, z \in \mathbb{U}). \tag{4}$$

This is called the generalized Bernardi integral operator. In terms of convolution, the operator $D^s_{a,\lambda}$ for $s \in \mathbb{C},\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \lambda > -1$ is defined as

$$D^s_{a,\lambda} = G^{(-1)}_{a,s}(z) * f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s \frac{(k+\lambda-1)!}{\lambda!\,(k-1)!} a_k z^k,$$

where $G^{(-1)}_{a,s}(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s \frac{(k+\lambda-1)!}{\lambda!(k-1)!} z^k$. For different choices of parameters, the operator $D^s_{a,\lambda}$ was extensively studied by various authors. For details, see [1, 6, 13, 14]. Furthermore, related to $D^s_{a,\lambda}$ the operator $I^s_{a,\lambda,\mu}$ can be defined as:

$$I^s_{a,\lambda,\mu} f(z) = \left(D^s_{a,\lambda}\right)^{(-1)}(z) * f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s \frac{(\mu)_{(k-1)}\, \lambda!}{(k+\lambda-1)!} a_k z^k, \tag{5}$$

where $s \in \mathbb{C},\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \lambda > -1, \mu > 0$ and $z \in U$. We see that $I^0_{0,0,2} f(z) = zf'(z)$ and $I^0_{0,1,2} f(z) = I^0_{0,0,2} f(z) = f(z)$. We also observe that $I^0_{0,\lambda,\mu}$ is introduced in [3] and $I^s_{a,0,\mu},\ \lambda \in \mathbb{R}, a > -1$ was studied by Cho et.al.[2]. Also $I^0_{0,2,\mu},\ \mu = 0, 1, 2, 3, ...$ is the Noor integral operator, see [8, 10]. From (5), we deduce the following

$$z\left(I^{s+1}_{a,\lambda,\mu} f(z)\right)' = (a+1)\, I^s_{a,\lambda,\mu} f(z) - a I^{s+1}_{a,\lambda,\mu} f(z), \tag{6}$$

$$z\left(I^s_{a,\lambda+1,\mu} f(z)\right)' = (\lambda+1)\, I^s_{a,\lambda,\mu} f(z) - \lambda I^s_{a,\lambda+1,\mu} f(z), \tag{7}$$

$$z\left(I^{s}_{a,\lambda,\mu}\,f(z)\right)' = \mu I^{s}_{a,\lambda,\mu+1}\,f(z) - (\mu-1)\,I^{s}_{a,\lambda,\mu}\,f(z), \tag{8}$$

and also from (1.4), we have

$$z\left(I^{s}_{a,\lambda,\mu}\,F_{\varsigma}(z)\right)' = (\varsigma+1)\,I^{s}_{a,\lambda,\mu}\,f(z) - \varsigma I^{s}_{a,\lambda,\mu}\,F_{\varsigma}(z). \tag{9}$$

Using the operator $I^{s}_{a,\lambda,\mu}$, we define the following subclasses of $\mathcal{A}$.

Definition 1.1. Let f be given by (1). Then $f \in \mathcal{S}^{s}(a,\lambda,\mu)$, if

$$q_0\,(z) = \frac{z\left(I^{s}_{a,\lambda,\mu}\,f(z)\right)'}{I^{s}_{a,\lambda,\mu}\,f(z)} \in \mathcal{P} \qquad (z \in \mathbb{U}), \tag{10}$$

where $s \in \mathbb{C},\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \lambda > -1, \mu > 0$ and $z \in U$.

Definition 1.2. Let f be given by (1). Then $f \in \mathcal{T}^{s}_{k}(a,\lambda,\mu,b)$, if there exists $g \in \mathcal{S}^{s}(a,\lambda,\mu)$ so that

$$1 + \frac{1}{b}\left(\frac{z\left(I^{s}_{a,\lambda,\mu}\,f(z)\right)'}{I^{s}_{a,\lambda,\mu}\,f(z)} - 1\right) \in \mathcal{P}_k \qquad (k \geq 2, z \in \mathbb{U}), \tag{11}$$

where $s \in \mathbb{C},\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \lambda > -1, \mu > 0, k \geq 2, b \in \mathbb{C}\backslash\{0\}$ and $z \in U$.

Various other known subclasses of analytic functions can be obtained for specific choices of the parameters s, a, λ and μ within their specific ranges, for example [2, 3, 8, 9, 10, 15, 16], with refrences therein.

2 Preliminary Notes

Lemma 2.1[5] If $h \in \mathcal{P}$, then $\frac{1-r}{1+r} \leq \Re h(z) \leq |h(z)| \leq \frac{1+r}{1-r}$ and $|h'(z)| \leq \frac{2\Re h(z)}{1-r^2}$.

Lemma 2.2[4] Let q be the convex in U and let $\omega : U \longrightarrow \mathbb{C}$, with $\mathrm{Re}\,\omega > 0$. If ϕ is analytic in U, then

$$\phi\,(z) + \omega(z) z\phi'\,(z) \prec q(z) \ \text{ implies } \ \phi\,(z) \prec q(z).$$

Lemma 2.3[7] Let ϑ be convex such that $\mathrm{Re}\,\{u\vartheta(z) + j\} > 0$ for $u, j \in C\backslash\{0\}$. If p is analytic with $p(0) = \vartheta(0)$ and $p(z) + \frac{zp'(z)}{up(z)+j} \prec \vartheta(z)$, then $p(z) \prec \vartheta(z)$.

Lemma 2.4 Let $s \in \mathbb{C},\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \lambda > -1, \mu > 0\ k \geq 2, b \in C\backslash\{0\}$ and $z \in U$. Then

$$(i)\mathcal{S}^{s}(a,\lambda,\mu) \subset \mathcal{S}^{s+1}(a,\lambda,\mu), \mathrm{Re}\,(a) \geq 0,$$

$$(ii)\ \mathcal{S}^{s}(a,\lambda,\mu) \subset \mathcal{S}^{s}(a,\lambda+1,\mu),\ \lambda \geq 0,$$

$$(iii)\ \mathcal{S}^{s}(a,\lambda,\mu+1) \subset \mathcal{S}^{s}(a,\lambda,\mu), \mu > 0.$$

Lemma 2.5 Let $f \in \mathcal{S}^s(a,\lambda,\mu)$, for $s \in \mathbb{C}, a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\lambda > -1, \mu > 0$ and $z \in U$. Then,$F_\varsigma \in \mathcal{S}^s(a,\lambda,\mu)$ where F_ς is given by (4).

Lemma 2.6 Let $s \in \mathbb{C}$, $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\lambda > -1, \mu > 0$ $k \geq 2, b \in C\backslash\{0\}$ and $z \in U$. Then

$$(i) \quad \mathcal{S}^{s+1}(a,\lambda,\mu) \subset \mathcal{S}^s(a,\lambda,\mu) \text{ for } \ |z| < r_0 = \frac{1+a_1}{2+\sqrt{3+a_1^2}}, \Re(a) = a_1 > 0,$$

$$(ii) \quad \mathcal{S}^s(a,\lambda+1,\mu) \subset \mathcal{S}^s(a,\lambda,\mu) \ \text{ for } \ |z| < r_0 = \frac{1+\lambda}{2+\sqrt{3+\lambda^2}}, \lambda \geq 0,$$

$$(iii) \quad \mathcal{S}^s(a,\lambda,\mu) \subset \mathcal{S}^s(a,\lambda,\mu+1) for\, |z| < r_0 = \frac{\mu}{2+\sqrt{3+(\mu-1)^2}}, \mu > 0.$$

Lemma 2.7*Let* $F_\varsigma \in \mathcal{S}^s(a,\lambda,\mu)$, where F_ς is given by (4). Then $f \in \mathcal{S}^s(a,\lambda,\mu)$ for $|z| < r_{0,}$ where $r_0 = \frac{1+\varsigma}{2+\sqrt{3+\varsigma^2}}$ and this result is sharp.

3 Results and Discussion

Theorem 2.1. Let $s \in \mathbb{C}$, $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\lambda > -1, \mu > 0$ $k \geq 2, b \in C\backslash\{0\}$ and $z \in U$. Then, $\mathcal{T}_k^s(a,\lambda,\mu,b) \subset \mathcal{T}_k^{s+1}(a,\lambda,\mu,b)$.

Proof. Let $f \in \mathcal{T}_k^s(a,\lambda,\mu,b)$. In view of Definition 1.2, there exists $g \in \mathcal{S}^s(a,\lambda,\mu)$ such that we have H as given in (11). Consider

$$q(z) = 1 + \frac{1}{b}\left(\frac{z\left(I_{a,\lambda,\mu}^{s+1} f(z)\right)'}{I_{a,\lambda,\mu}^{s+1} g(z)} - 1\right). \tag{12}$$

For $g \in \mathcal{S}^s(a,\lambda,\mu)$, this follows by Lemma 2.4(i) that $g \in \mathcal{S}^{s+1}(a,\lambda,\mu)$. Furthermore, using (6), (10) (12) and then simplifying, we have

$$H(z) = q(z) + \frac{zq'(z)}{q_0(z)+a} \in \mathcal{P}_k. \tag{13}$$

Consider that

$$q(z) = \left(\frac{k}{4}+\frac{1}{2}\right)q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)q_2(z). \tag{14}$$

On combining (13) and (14), we deduce

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z), \tag{15}$$

where $H_i(z) = q_i(z) + \frac{zq_i'(z)}{q_0(z)+a}, i = 1,2$. From (13) and (14), we write $H_i \in \mathcal{P}$, $i = 1,2$. For $\mathrm{Re}\,(q_0(z)+a) > 0$, this result along with Lemma 2.3 imply that $q_i \in \mathcal{P}$, $i = 1,2$ and $z \in U$. Thus from (13), we have $f \in \mathcal{T}_k^{s+1}(a,\lambda,\mu,b)$.

Theorem 2.2. Let $s \in \mathbb{C},\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \lambda > -1, \mu > 0\ k \geq 2, b \in C\backslash\{0\}$ and $z \in U$. Then,$\mathcal{T}_k^s(a, \lambda, \mu, b) \subset \mathcal{T}_k^s(a, \lambda + 1, \mu, b)$.

Following procedures of the above theorem along with (2), the identity (8), (10), (11), Lemma 2.3 and Lemma 2.4(iii), we obtain the desired proof.

Theorem 2.3. Let $s \in \mathbb{C},\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \lambda > -1, \mu > 0\ k \geq 2, b \in C\backslash\{0\}$ and $z \in U$. Then,$\mathcal{T}_k^s(a, \lambda, \mu, b) \subset \mathcal{T}_k^s(a, \lambda + 1, \mu, b)$.

Proof. Let $f \in \mathcal{T}_k^s(a, \lambda, \mu, b)$. Then by(11), we have (12). Consider

$$q(z) = 1 + \frac{1}{b}\left(\frac{z\left(I_{a,\lambda+1,\mu}^s\, f(z)\right)'}{I_{a,\lambda+1,\mu}^s\, g(z)} - 1\right). \tag{16}$$

For $g \in \mathcal{S}^s(a, \lambda, \mu)$, this follows by Lemma 2.4 that $g \in \mathcal{S}^s(a, \lambda + 1, \mu)$. Furthermore, using (10), (7) and (16) and then simplifying, we have

$$H(z) = \left(q(z) + \frac{zq'(z)}{q_0(z) + \lambda}\right) \in P_k. \tag{17}$$

On combining (17) and (14), we obtain

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) H_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) H_2(z). \tag{18}$$

From (17) and (18), we deduce that $H_i(z) = q_i(z) + \frac{zq_i'(z)}{q_0(z)+\lambda} \in P, i = 1, 2$. For $\mathrm{Re}(q_0(z) + \lambda) > 0$, Lemma 2.3 imply that $q_i \in P$, $i = 1$, 2 and $z \in U$. Hence $T_k^s(a, \lambda, \mu, b) \subset T_k^s(a, \lambda + 1, \mu, b)$▲.

Theorem 3.4. Let $s \in C, a \in C\backslash Z_0^-, \lambda > -1, k \geq 2, \mu > 0, b \in C\backslash\{0\}$ and $z \in U$. Then $T_k^s(a, \lambda, \mu + 1, b) \subset T_k^s(a, \lambda, \mu, b)$.

Proof. Let $f \in T_k^s(a, \lambda, \mu, b)$. Consider

$$q(z) = 1 + \frac{1}{b}\left(\frac{z\left(I_{a,\lambda,\mu}^s\, F_\zeta(f)(z)\right)'}{I_{a,\lambda,\mu}^s\, F_\zeta(g)(z)} - 1\right) \tag{19}$$

For $g \in S^s(a, \lambda, \mu)$, it follows from Lemma 2.5 that $F_\zeta(g) \in S^s(a, \lambda, \mu)$. Furthermore, for $F_\zeta(g) \in S^s(a, \lambda, \mu)$, by Definition 1.1, (10) and (19), we have

$$H(z) = \left(q(z) + \frac{zq'(z)}{q_0(z) + \zeta}\right) \in P_k. \tag{20}$$

On combining (14) and (20), we obtain

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) H_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) H_2(z). \tag{21}$$

From (14) and (21), we deduce that

$$H_i(z) = q_i(z) + \frac{zq_i'(z)}{q_0(z) + \zeta} \in P, i = 1, 2. \tag{22}$$

For $\text{Re}(q_0(z)+\zeta)>0$, Lemma 2.3 and (22) imply that $q_i \in P$, $i=1,2$. Thus from (20), we have $F_\zeta(f) \in T_k^s(a,\lambda,\mu,b)$.

Our next theorems, we deal with radii problems $f \in T_k^s(a,\lambda,\mu,b)$.

Theorem 3.5. For $|z| < r_0 = \frac{1+a_1}{2+\sqrt{3+a_1^2}}$, $T_k^{s+1}(a,\lambda,\mu,b) \subset T_k^s(a,\lambda,\mu,b)$. The value of r_0 is sharp.

Proof. Let $f \in T_k^{s+1}(a,\lambda,\mu,b)$. Then by Definition 1.2, there exists $g \in S^{s+1}(a,\lambda,\mu)$ such that we have H as in (11). We take

$$q(z) = 1 + \frac{1}{b}\left(\frac{z\left(I_{a,\lambda,\mu}^s\, f(z)\right)'}{I_{a,\lambda,\mu}^s\, g(z)} - 1\right). \tag{23}$$

For $g \in S^{s+1}(a,\lambda,\mu)$, from Lemma 2.6(i), we have $g \in S^s(a,\lambda,\mu)$. Furthermore, using (6), (10) and (23), we have

$$H(z) = q(z) + \frac{zq'(z)}{q_0(z)+a} \in P_k. \tag{24}$$

On combining (14) and (24), we obtain

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right) H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) H_2(z). \tag{25}$$

where $H_i(z) = q_i(z) + \frac{zq_i'(z)}{q_0(z)+a}$, $i=1,2$ and $z \in U$. To find that $H_i \in P$, for $i=1,2$ and $z \in U$, we use Lemma 2.1, we have

$$\Re\left(q_i(z) + \frac{zq_i'(z)}{q_0(z)+a}\right) \geq \Re q_i(z)\left(\frac{(1-a_1)r^2 - 4r + 1 + a_1}{((1-r)^2 + (1-r^2)a_1)}\right) \tag{26}$$

The inequality (26) is positive if $|z| < r_0 = \frac{1+a_1}{2+\sqrt{3+a_1^2}}$ and the sharpness of r_0 follows from $q_i = p_0$ given by (2). Hence, from (24) and (25), we conclude the desire result.

Theorem 3.6. For $|z| < r_0 = \frac{1+\lambda}{2+\sqrt{3+\lambda^2}}$, $T_k^s(a,\lambda+1,\mu,b) \subset T_k^s(a,\lambda,\mu,b)$. The value of r_0 is sharp.

Using (2), the identity (7), (10), (11), Lemma 2.1and Lemma 2.6 (ii), we obtain the desired proof. □

Theorem 3.7. For $|z| < r_0 = \frac{1+(\mu-1)}{2+\sqrt{3+(\mu-1)^2}}$, $T_k^s(a,\lambda,\mu+1,b) \subset T_k^s(a,\lambda,\mu,b)$. The value of r_0 is sharp.

Following similar procedure as in Theorem 3.5, using (2) the identity (8), (10), (11), Lemma 2.1 and Lemma 2.6 (iii), we obtain the required proof. □

Theorem 3.8. For $F_\zeta \in T_k^s(a,\lambda,\mu)$, $f \in T_k^s(a,\lambda,\mu)$ for $|z| < r_0 = \frac{1+\zeta}{2+\sqrt{3+\zeta^2}}$ and this result is sharp.

Proof. Let $F_\zeta \in T_k^s(a,\lambda,\mu)$. Consider

$$q\left(z\right) = 1 + \frac{1}{b}\left(\frac{z\left(I_{a,\lambda,\mu}^s\, f(z)\right)'}{I_{a,\lambda,\mu}^s\, g(z)} - 1\right). \tag{27}$$

For $F_\zeta(g) \in S^s(a,\lambda,\mu)$, Lemma 2.7 yields $g \in S^s(a,\lambda,\mu)$. Furthermore, for $F_\zeta(g) \in S^s(a,\lambda,\mu)$, by Definition 1.1, (9) and (27), we have

$$H(z) = q(z) + \frac{zq'(z)}{q_0(z)+\zeta} \in P_k. \tag{28}$$

On combining (14) and (28), we obtain

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z). \tag{29}$$

where $H_i(z) = q_i(z) + \frac{zq_i'(z)}{q_0(z)+\zeta}, i = 1,2$ and $z \in U$. To find $H_i \in P$, for $i = 1,2$ and $z \in U$, we use Lemma 2.1, we have

$$\Re\left(q_i(z) + \frac{zq_i'(z)}{q_0(z)+\zeta}\right) \geq \Re q_i(z)\frac{(1-\zeta)r^2 - 4r + 1 + \zeta}{(1-r)^2 + (1-r^2)\zeta}. \tag{30}$$

The inequality (30) is positive if $|z| < r_0 = \frac{1+\zeta}{2+\sqrt{3+\zeta^2}}$ and the sharpness of r_0 follows from $q_i = p_0$ given by (2). Hence, from (28) and (29), we conclude the desire result.

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