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Some Properties of the Essential Numerical Range on Banach Spaces

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Abstract

We study the properties of the essential algebraic numerical range as well as the essential spatial numerical range for Banach space operators.

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1 Introduction

Let $\mathcal{A}$ be a complex normed algebra with a unit and let $\mathcal{A}^*$ denote its dual space. For a comprehensive theory on normed algebras, we refer to [4]. We define the algebraic numerical range of an element $a \in \mathcal{A}$ by $V(a, A) = \{f(a) : f \in \mathcal{A}^*, f(1) = 1 = \|f\|\}$. Let X denote a complex Banach space and $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators acting on X. We denote the algebraic numerical range of any $T \in \mathcal{L}(X)$ by $V(T, \mathcal{L}(X))$. For $T \in \mathcal{L}(X)$, the spatial numerical range is defined by $W(T) = \{\langle Tx, x^* \rangle : x \in X, x^* \in X^*, \|x\| = 1 = \|x^*\| = \langle x, x^* \rangle\}$. In the case that X is a Hilbert space the definition reduces to $W(T) = \{\langle Tx, x \rangle \; x \in X, \|x\| = 1\}$ which is the well known definition of the Hilbert space numerical range.
The algebraic and spatial numerical ranges of any $T \in \mathcal{L}(X)$ are closely related and this relationship has been a problem of study for the past few decades. See for instance [3, 4, 7, 8, 9] and references therein. In particular, for Hilbert

space operators the set $W(T)$ is convex by the classical Toeplitz-Hausdorff theorem and $V(T, \mathcal{L}(X)) = \overline{W(T)}$ where $\overline{W(T)}$ is the closure of $W(T)$. The above assertions are no longer true for Banach space operators. Specifically we only have $V(T, \mathcal{L}(X)) = \overline{conv} W(T)$, where $conv$ denotes the convex hull, see [3, 5] for details.
Let $\mathcal{K}(X)$ denote the ideal of all compact operators acting on a complex Banach space X. Also let q be the usual canonical mapping from $\mathcal{L}(X)$ onto the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. The essential algebraic numerical range, $V_e(T)$, of an operator $T \in \mathcal{L}(X)$ where X is an infinite dimensional Banach space is defined by $V_e(T) = V(q(T), \mathcal{L}(X)/\mathcal{K}(X), \|.\|_e)$ where $\|.\|_e$ denotes the essential norm given by $\|T\|_e = \inf\{\|T + K\| : K \in \mathcal{K}(X)\}$. On the other hand the essential spatial numerical range $W_e(T)$ of an operator $T \in \mathcal{L}(X)$ is defined to be the set of all complex numbers λ with the property that there are nets $(u_\alpha) \subset X, (u_\alpha^*) \subset X^*$, such that $\|u_\alpha\| = \|u_\alpha^*\| = \langle u_\alpha, u_\alpha^* \rangle = 1$ for all α, $u_\alpha \to 0$ weakly and $\langle Tu_\alpha, u_\alpha^* \rangle \to \lambda$.
As remarked in [2], the essential algebraic and the essential spatial numerical ranges for a Hilbert space operators coincide. For Banach space operators, the properties of the essential algebraic numerical range have been remarkably studied in literature. Surprisingly, the first reasonable attempt for the corresponding study of the essential spatial numerical range was by [2] in 2005. The reasons behind this strange observation isn't apparent but we believe it might be due to the lack of the equality in $W_e(T) \neq W(q(T))$. It is also noted in [2] that for a successful study of the properties of this numerical range, it is important to consider another norm which is a "measure of non-compactness" instead of the usual essential norm $\|.\|_e$. In this study we consider these two numerical ranges on Banach spaces. Apart from extending the well known properties of the essential algebraic numerical range, we establish some new properties of the spatial numerical range.

2 Essential Algebraic Numerical Range

Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. The basic properties of the essential algebraic numerical range $V_e(T)$ can be found in [5] and we summarize them in the following theorem;

Theorem 2.1. *(i)* *$V_e(T)$ is a nonempty compact subset of $\mathbb{C}$ and $\sigma_e(T) \subset V_e(T)$ where $\sigma_e(T)$ denotes the essential spectrum of T.*

(ii) *$V_e(T) = \{0\}$ if and only if $T \in \mathcal{K}(X)$.*

(iii) *$V_e(T) = \bigcap\{V(T + K, B(X)) : K \in \mathcal{K}(X)\}$.*

(iv) *$V_e(T) = \{f(T) : f \in B(X)^*, f(I) = 1 = \|f\|, f(\mathcal{K}(X)) = \{0\}\}$*

(v) $exp(-1).\|T\|_e \leq max\{|\lambda| : \lambda \in V_e(T)\} \leq \|T\|_e$.

As an extension of the above properties, we establish the following additional algebraic properties of $V_e(T)$.

Theorem 2.2. *For $T, S \in \mathcal{L}(X)$ and $\alpha,\ \beta \in \mathbb{C}$, we have:*

(i) $V_e(\alpha T) = \alpha V_e(T)$

(ii) $V_e(T + S) \subseteq V_e(T) + V_e(S)$.

(iii) $V_e(\alpha T + \beta S) \subseteq \alpha V_e(T) + \beta V_e(S)$.

Proof. To prove (i), let p be a complex number. Then $p \in V_e(T)$ if and only if $|p-\lambda| \leq \|T+K-\lambda\|$ for each complex number λ and each compact operator K. So $p \in V_e(\alpha T)$ if and only if $|p-\lambda| \leq \|\alpha((T+K)-\lambda)\| = |\alpha|\|T+K-\lambda\|$ for each complex number α and λ and each compact operator K. Hence $V_e(\alpha T) = V(q(\alpha T) = V(\alpha q(T)) = \alpha V(q(T)) = \alpha V_e(T)$.
Now, $V_e(T+S) = V(q(T+S)) = V((T+S)+K) = V((T+K)+(S+K)) \subseteq V(T+K) + V(S+K) = V(q(T)) + V(q(S)) = V_e(T) + V_e(S)$, which proves (ii).
The proof of (iii) follows from (i) and (ii) above. □

Let $\nu_e(T)$ denotes the essential algebraic numerical radius defined by $\nu_e(T) = \sup\{|\lambda| :\ \lambda \in V_e(T)\}$, that is, the numerical radius associated with $V_e(T)$, then we obtain the following consequence of the above theorem;

Corollary 2.3. *For $T, S \in \mathcal{L}(X)$ and α,$\beta \in \mathbb{C}$, we have:*

(i) $\nu_e(\alpha T) = |\alpha|\nu_e(T)$

(ii) $\nu_e(T + S) \leq \nu_e(T) + \nu_e(S)$

(iii) $\nu_e(\alpha I + T) = |\alpha| + \nu_e(T)$

(iv) $\nu_e(\alpha T + \beta S) \leq |\alpha|\nu_e(T) + |\beta|\nu_e(S)$.

Proof. Follows immediately from Theorem 2.2 and the definition of $\nu_e(T)$. □

For compact operators, the following theorem details the relation between the essential spectrum and the essential algebraic numerical range;

Theorem 2.4. *If $T, S \in \mathcal{K}(X)$ and $\alpha \in \mathbb{C}$, then*

(i) $\sigma_e(T) = V_e(T) = \{0\}$

(ii) $\sigma_e(T + S) = V_e(T + S) = V_e(T) + V_e(S)$.

(iii) $\sigma_e(T^*) = V_e(T^*)$

(iv) $\sigma_e(\alpha T) = V_e(\alpha T)$

Proof. For $T \in \mathcal{K}(X)$, $V_e(T) = \{0\}$. But $\sigma_e(T) \subseteq V_e(T)$, and since the spectrum $\sigma_e(T)$ is nonempty, the result follows, and this proves (i). To prove (ii), since $T, S \in \mathcal{K}(X)$, $T + S \in \mathcal{K}(X)$ and $\{0\} = \sigma_e(T + S) \subseteq V_e(T + S) = V_e(T) + V_e(S) = \{0\}$.
Assertion (iii) follows from the fact that $\sigma_e(T) = \sigma_e(T^*)$, while assertion (iv) is obvious. □

Consequently, if $r_e(T)$ is the essential spectral radius in the sense that $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$, then we have the following result;

Corollary 2.5. *If* $T \in \mathcal{K}(X)$,*then* $r_e(T) = \nu_e(T) = 0$

3 Essential Spatial Numerical Range

The literature on the study of essential spatial numerical range for Banach space operators is very scanty. One known study that's available in literature is the work [2] by Barraa and Müller. In their attempt to study some properties of $W_e(T)$ on Banach spaces, the authors in [2] considered another measure of non-compactness instead of the essential norm in the Calkin algebra. They remarked that this is a probable reason why $W_e(T)$ has never been studied before. For $T \in \mathcal{L}(X)$ where X is an infinite dimensional Banach space, we define a seminorm $\|.\|_\mu$ on $\mathcal{L}(X)$ by $\|T\|_\mu = \inf\{\|T|_M\| : M \subset X$ a subspace of finite codimension $\}$. Following [2], $\|.\|_\mu$ is a measure of non-compactness, that is, $\|T\|_\mu = 0$ if and only if T is compact. Moreover, $\|.\|_\mu$ is an algebra norm on the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. As a result, another essential numerical range $V_\mu(T)$ defined by $V_\mu(T) = V(T, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu)$ was introduced. In particular, $V_\mu(T)$ is the set of all $\lambda \in \mathbb{C}$ such that there is a functional $\tilde{\Phi} \in (\mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu)^*$ satisfying $\|\tilde{\Phi}\| = 1 = \tilde{\Phi}(I + \mathcal{K}(X))$ and $\tilde{\Phi}(T + \mathcal{K}(X)) = \lambda$. Equivalently, there is a functional $\Phi \in \mathcal{L}(X)^*$ such that $\Phi(\mathcal{K}(X) = 0, \Phi(I) = 1, \Phi(T) = \lambda$ and $|\Phi(S)| \leq \|S\|_\mu$ for all $S \in \mathcal{L}(X)$.
Before looking at the properties of the essential spatial numerical range $W_e(T)$, we summarize some properties of $V_\mu(T)$ in the following theorem:

Theorem 3.1. *Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. Then*

(i) $V_\mu(T)$ *is a closed, convex and compact subset of* $\mathbb{C}$.

(ii) $V_\mu(T) = \{0\}$ *if and only if* T *is compact*

(iii) $V_\mu(T + K) = V_\mu(T)$ *for* $K \in \mathcal{K}(X)$.

(iv) $V_\mu(T + S) \subseteq V_\mu(T) + V_\mu(S)$ *where* $S \in \mathcal{L}(X)$.

Proof. To prove that $V_\mu(T)$ closed, let $\lambda_n \in V_\mu(T)$ be such that $\lambda_n \to \lambda$ as $n \to \infty$. Then for a fixed n, there exists $\Phi_n \in \mathcal{L}(X)^*$ such that $\Phi_n(\mathcal{K}(X)) = 0$, $\Phi_n(I) = 1$, $\Phi_n(T) = \lambda_n$ and $\|\Phi_n(S)\| \leq \|S\|_\mu$ for all S. Then $\Phi(I) = \lim_n \Phi_n(I) = 0$, $\Phi(T) = \lim_n \Phi_n(T) = \lim_n \lambda_n = \lambda$, $\Phi(\mathcal{K}(X)) = \lim_n \Phi_n(\mathcal{K}(X)) = 0$ and $\|\Phi(S)\| = \lim_n \|\Phi_n(S)\| \leq \|S\|_\mu$ for all S. Thus $\lambda \in V_\mu(T)$ and this proves that $V_\mu(T)$ is closed. Following [2], $V_\mu(T) = conv(W_e(T))$ which clearly indicates that $V_\mu(T)$ is convex since it is a convex hull of some set. In general, we know that $V_\mu(T) \subset V_e(T)$. But $V_e(T)$ is compact [5] and $V_\mu(T)$ is closed . The result then follows immediately from the fact that a closed subset of a compact set is compact. This proves (i).
Now, for any $T \in \mathcal{L}(X)$, we have that $V_\mu(T) = \{0\}$ if and only if $V_e(T) = \{0\}$ which is true if and only if T is compact. This proves (ii).
The proof of (iii) follows from the definition of $V_\mu(T)$ and from the fact that $\|.\|_\mu$ is a measure of non-compactness.
For (iv), from the sum property of the algebraic numerical range, we have

$$\begin{aligned} V_\mu(T+S) &= V(T+S, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu) \\ &\subseteq V(T, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu) + V(S, \mathcal{L}(X)/\mathcal{K}(X), \|.\|_\mu) \\ &= V_\mu(T) + V_\mu(S). \end{aligned}$$

□

Define $\nu_\mu(T) = \sup\{|\lambda| : \lambda \in V_\mu(T)\}$ as the numerical radius corresponding to the numerical range $V_\mu(T)$. We can then deduce the following Corollary;

Corollary 3.2. *Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. Then*

(i) $\nu_\mu(T) = \{0\}$ *if and only if* T *is compact*

(ii) $\nu_\mu(T+K) = \nu_\mu(T)$ *for* $K \in \mathcal{K}(X)$

(iii) $\nu_\mu(T+S) \subseteq \nu_\mu(T) + \nu_\mu(S)$ *where* $S \in \mathcal{L}(X)$.

(iv) $\nu_\mu(T^*) = \nu_\mu(T)$.

Proof. Follows from the definition of $\nu_\mu(T)$ and Theorem 3.1 above. □

It's important to take note that in general $V_\mu(T) \subset V_e(T)$, but if X is a Hilbert space we obtain equality, that is, $V_\mu(T) = V_e(T)$.
The next result gives some properties of the essential spatial numerical range.

Theorem 3.3. *For $T \in \mathcal{L}(X)$, we have*

(i) $W_e(T)$ *is nonempty closed non-convex and compact subset of the complex plane* $\mathbb{C}$.

(ii) $W_e(T) = \{0\}$ *if and only if* T *is compact*

(iii) $W_e(\beta T) = \beta W_e(T)$ *for some* $\beta \in \mathbb{C}$.

(iv) $W_e(T + S) \subseteq W_e(T) + W_e(S)$, *where* $S \in \mathcal{L}(X)$.

(v) $W_e(\alpha T + \beta S) \subseteq \alpha W_e(T) + \beta W_e(S)$, *where* $S \in \mathcal{L}(X)$ *and* β, $\alpha \in \mathbb{C}$.

Proof. Following [2], $\sigma_e(T) \subset W_e(T)$. Since $\sigma_e(T)$ is nonempty, it follows that $W_e(T)$ is nonempty as well. The noncovexity of $W_e(T)$ is immediate from the relation $V_\mu(T) = conv(W_e(T))$. For closedness, let $\lambda_n \in W_e(T)$ be such that $\lambda_n \to \lambda$ as $n \to \infty$. We want to show that $\lambda \in W_e(T)$. Since $\lambda_n \in W_e(T)$, choose nets which are partially ordered on subsets of X and X^* by the relation $\leq$ as $(u_\alpha) \subset X$, $(u^*_\alpha) \subset X^*$ such that $\|u_\alpha\| = \|u^*_\alpha\| = \langle u_\alpha, u^*_\alpha\rangle = 1$ for each α and $u_\alpha \to 0$ weakly. Fix n such that $|\langle Tu_\alpha, u^*_\alpha\rangle - \lambda_n| < \frac{1}{n}$. Then

$$\begin{aligned} |\langle Tu_\alpha, u^*_\alpha\rangle - \lambda| &\leq |\langle Tu_\alpha, u^*_\alpha\rangle - \lambda_n| + |\lambda_n - \lambda| \\ &< \tfrac{1}{n} + |\lambda_n - \lambda| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

The compactness of $W_e(T)$ follows from the compactness of $V_\mu(T)$ since $W_e(T)$ is a closed subset of $V_\mu(T)$. This proves (i).
To prove (ii), take note that $W_e(T) \subset conv(W_e(T)) = V_\mu(T) = \{0\}$ if and only if T is compact. Since $W_e(T)$ is nonempty, the latter statement is equivalent to $W_e(T) = \{0\}$ if and only if T is compact, as desired.
For (iii), let $\lambda \in W_e(T)$. This is equivalent to $\exists\, (u_\alpha) \subset X$, $(u^*_\alpha) \subset X^*$ such that $\|u_\alpha\| = \|u^*_\alpha\| = \langle u_\alpha, u^*_\alpha\rangle = 1$ for all $\alpha, u_\alpha \to 0$ weakly and $\langle Tu_\alpha, u^*_\alpha\rangle \to \lambda$. Then $\beta W_e(T) \Leftrightarrow \beta\langle Tu_\alpha, u^*_\alpha\rangle \to \beta\lambda$. This in turn is equivalent to $\langle \beta Tu_\alpha, u^*_\alpha\rangle \to \beta\lambda$. Since $\beta\lambda \in \mathbb{C}$, it follows that $\beta W_e(T) = W_e(\beta T)$.
To prove (iv), let $\lambda \in W_e(T + S)$. Then $\exists\, (u_\alpha) \subset X$, $(u^*_\alpha) \subset X^*$ such that $\|u_\alpha\| = \|u^*_\alpha\| = \langle u_\alpha, u^*_\alpha\rangle = 1$ for all $\alpha, u_\alpha \to 0$ weakly and $\langle (T+S)u_\alpha, u^*_\alpha\rangle \to \lambda$. Then $\langle Tu_\alpha + Su_\alpha, u^*_\alpha\rangle \to \lambda$ is equivalent to $\langle Tu_\alpha, u^*_\alpha\rangle + \langle Su_\alpha, u^*_\alpha\rangle \to \lambda$. This implies that $\langle Tu_\alpha, u^*_\alpha\rangle \to \lambda_1$ and $\langle Su_\alpha, u^*_\alpha\rangle \to \lambda_2$ where $\lambda = \lambda_1 + \lambda_2$. Thus, $\lambda_1 \in W_e(T)$ and $\lambda_2 \in W_e(S)$ with $\lambda = \lambda_1 + \lambda_2$. Hence $\lambda \in W_e(T) + W_e(S)$.
Assertion (v) follows immediately from assertions (iii) and (iv) above. □

Define the essential spatial numerical radius of T, $\omega_e(T)$, by

$$\omega_e(T) = \sup\{|\lambda| : \lambda \in W_e(T)\}.$$

Then the following is an immediate consequence of Theorem 3.3 above.

Corollary 3.4. *For* $T, S \in \mathcal{L}(X)$ *and* α, $\beta \in \mathbb{C}$, *we have*

(i) $\omega_e(T) = 0$ *if and only if* $T \in \mathcal{K}(X)$

(ii) $\omega_e(\beta T) = |\beta|\omega_e(T)$

(iii) $\omega_e(T + S) \leq \omega_e(T) + \omega_e(S)$

(iv) $\omega_e(\alpha T + \beta S) \leq |\alpha|\omega_e(T) + |\beta|\omega_e(S)$.

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