Pure Mathematical Sciences, Vol. 6, 2017, no. 1, 95 - 104
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/pms.2017.7911

On (f, g)-Derivations of G-Algebras

Deena Al-Kadi

Department of Mathematics and Statistic
Taif University
Taif, Saudi Arabia

Abstract

The notion of an (f, g)-derivation of a G-algebra is introduced and some related properties are investigated. Also the concept of regular (f, g)-derivation is provided and some results are obtained. Moreover, a condition of two (f, g)-derivations to be an (f, g)-derivation is given.

Mathematics Subject Classification: 03G25, 06F35

Keywords: Derivations, G-algebra

1 Introduction and preliminaries

The notion of derivations as defined in rings and near-rings theory (see [6]) has been applied to BCI-algebra by Jun and Xin in [7] and then by Abujabal and Al-Shehri in [1]. Many researches have been done on derivations of BCI-algebra in different aspects. For example, (α, β)-derivations of BCI-algebra was introduced in [9] and some related properties were investigated. In [10], the notion of t-derivations of BCI-algebra was given and the study was extended to t-derivations of a p-semisimple BCI-algebra. In [11], the notion of (θ, ϕ)-derivations of a BCI-algebra and some results on inside (or outside) (θ, ϕ)-derivations of BCI-algebra are discussed. Then a new kind of derivations of BCI-algebra has been introduced in [8]. On the other hand, some authors applied the notion of derivations on different classes of abstract algebras such as BCC-algebra, B-algebra and G-algebra and some related properties have been investigated (see [12], [4] and [3].) Moreover, a new kind of derivations of G-algebra named f_q-derivation has been introduced in [2].

In this paper we continue to study derivations of G-algebra, in particular, (f, g)-derivation. We start, in Section 1, by giving definitions and propositions needed. In Section 2, we introduce the notion of an (f, g)-derivation and regular (f, g)-derivation of a G-algebra and we obtain some results. Moreover, we give a condition for the composition of two (f, g)-derivations of a G-algebra to be an (f, g)-derivation. We recall the following definitions and propositions of G-algebra.

Definition 1.1 *([5, Definition 2.1]) A G-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the axioms:*

(1) $x * x = 0$,

(2) $x * (x * y) = y$, *for all* $x, y \in X$.

Proposition 1.2 *([5, Proposition 2.1]) If $(X, *, 0)$ is a G-algebra, then the following conditions hold:*

(1) $x * 0 = x$,

(2) $0 * (0 * x) = x$, *for any* $x \in X$.

Proposition 1.3 *([5, Proposition 2.2]) Let $(X, *, 0)$ be a G-algebra. Then, the following conditions hold for any $x, y \in X$,*

(1) $(x * (x * y)) * y = 0$,

(2) $x * y = 0 \Longrightarrow x = y$,

(3) $0 * x = 0 * y \Longrightarrow x = y$.

Theorem 1.4 *([5, Theorem 2.6]) Let X be a G-algebra. Then the following are equivalent, for all $x, y, z \in X$:*

(1) $(x * y) * z = (x * z) * y$.

(2) $(x * y) * (x * z) = z * y$.

Theorem 1.5 *([5, Theorem 2.7]) Let X be a G-algebra. Then the following are equivalent, for all $x, y, z \in X$:*

(1) $(x * y) * (x * z) = (z * y)$.

(2) $(x * z) * (y * z) = x * y$.

Lemma 1.6 *([5, Lemma 2.1]) Let $(X, *, 0)$ be a G-algebra. Then $a*x = a*y$ implies $x = y$ for any $a, x, y \in X$.*

Definition 1.7 *([5, Definition 3.2]) For any G-algebra X, the set $B(X) = \{x \in X \mid 0 * x = 0\}$ is called a p-radical of X. If $B(X) = \{0\}$ then G-algebra is said to be p-semisimple. The set $G(X) = \{x \in X \mid 0 * x = x\}$ is called the G-part of X. It is obvious that $B(X) \bigcap G(X) = \{0\}$.*

Proposition 1.8 *([5, Proposition 3.1]) Let X be a G-algebra. Then $x \in G(X)$ if and only if $0 * x \in G(X)$.*

Theorem 1.9 *([5, Theorem 3.4]) Let X be a G-algebra. If $G(X) = X$ then X is p-semisimple.*

Definition 1.10 *([5, Definition 3.3]) A G-algebra $(X, *, 0)$ satisfying:*

$$(x * y) * (z * u) = (x * z) * (y * u), \text{ for any } x, y, z, u \in X$$

is called a medial G-algebra.

Lemma 1.11 *([5, Lemma 3.1]) If X is a medial G-algebra, then for any $x, y, z \in X$, the following holds:*

(1) $(x * y) * x = 0 * y$,

(2) $x * (y * z) = (x * y) * (0 * z)$,

(3) $(x * y) * z = (x * z) * y$.

Definition 1.12 *([5, Definition 3.1]) A G-algebra X is said to be 0-commutative if $x * (0 * y) = y * (0 * x)$, for all $x, y \in X$.*

In G-algebra, define the binary operation $\wedge$ of the elements x and y as $x \wedge y = y * (y * x)$, for all $x, y \in X$. An element $a \in X$ is said to be an initial element (p-atom) of X, if $x * a = 0$ implies $x = a$. Denote the set of all initial elements in X by $L_p(X)$. Note that $L_p(X) = \{a \in X \mid 0 * (0 * a) = a\}$.

Define $-$ operation as $x - y = x * y$, for all $x, y \in X$ and define $+$ as $x + y = x * (0 * y)$, for all $x, y \in X$. Then in any G-algebra $(X, *, 0)$ the following is true for all $x, y \in X$:

(1) $x + 0 = 0 + x = x$,

(2) $(x + y) + z = x + (y + z)$.

Table 1:

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Table 2:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Definition 1.13 *([3, Definition 3.1]) Let X be a G-algebra and d a self-map of X. We say that d is a left-right derivation (briefly (l, r)-derivation) of X if it satisfies the identity $d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$. If d satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$, then d is said to be a right-left derivation (briefly (r, l)-derivation) of X. If d is both (l, r)-and (r, l)-derivation, then d is a derivation of X.*

Example 1.14 *Consider the G-algebra given by Cayley Table 1. Define a map $d : X \longrightarrow X$ by:*

$$d(x) = \begin{cases} 1, & \textit{if } x \textit{ in } 0,\ 2; \\ 0, & \textit{otherwise.} \end{cases}$$

Then by direct calculations we have $d(2*2) = d(0) = 1$ and $d(2)*2 = 1*2 = 2$. As $d(2 * 2) \neq d(2) * 2$ then d is not (l, r)-derivation of X. Similarly, d is not (r, l)-derivation of X as $d(1 * 2) \neq 1 * d(2)$.

Example 1.15 *Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by Table 2. Define a map $d : X \longrightarrow X$ by:* $d(x) = \begin{cases} 2, & \textit{if } x{=}0; \\ 3, & \textit{if } x{=}1; \\ 0, & \textit{if } x{=}2; \\ 1, & \textit{if } x{=}3. \end{cases}$

Define a map $d : X \longrightarrow X$ by:

Then it is straight forward to check that d is a derivation of X.

Table 3:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

2 The (f, g)-derivation of G-algebras

In this section we start with the notion of an (f, g)-derivation of a G-algebra, where f and g are endomorphisms of a G-algebra for the rest of the paper unless otherwise specified.

Definition 2.1 *Let X be a G-algebra. A left-right (f, g)-derivation (briefly (l, r)-(f, g)-derivation) of X is a self-map d of X satisfying the following identity, for all $x, y \in X$:*

$$d(x * y) = (d(x) * f(y)) \wedge (g(x) * d(y)).$$

If d satisfies the identity:

$$d(x * y) = (f(x) * d(y)) \wedge (d(x) * g(y)),$$

then d is a right-left (f, g)-derivation (briefly (r, l)-(f, g)-derivation) of X. If d is both (l, r)-(f, g)-derivation and (r, l)-(f, g)-derivation of X then we say that d is an (f, g)-derivation of X.

Example 2.2 *Consider the G-algebra $(X, *, 0)$ given by Cayley Table 3.*

Let d be the zero map and f and g endomorphisms defined as follow: $f(x) = 0$ and

$$g(x) = \begin{cases} 0, & \textit{if } x = 0; \\ 2, & \textit{if } x = 1; \\ 1, & \textit{if } x = 2. \end{cases}$$

Then, by using the assumption that d and f are zero maps, it is direct to show that d is an (f, g)-derivation of X.

This leads to the following result:

Corollary 2.3 *Let X be a G-algebra and f be the zero map. If d is the zero map then d is an (f, g)-derivation of X.*

Example 2.4 *Consider Example 2.2 with f an identity endomorphism. As d is the zero map, we have $d(x * y) = 0$ but $d(x) * f(y) = 0 * y \neq 0$ for some $y \in X$. Therefore, d is not a (l, r)-(f, g)-derivation of X neither a (r, l)-(f, g)-derivation of X as $f(x) * d(y) = x * 0 = x \neq 0$, for all $x \in X \backslash \{0\}$.*

Remark: In G-algebra, if f and g are the identity maps then every (f, g)-derivation is the derivation defined in Definition 1.13.

Proposition 2.5 *Let d be a (l, r)-(f, g)-derivation of a medial G-algebra X. Then $d(x) = d(x) \wedge g(x)$, for all $x \in X$.*

Proof: Let d be a (l, r)-(f, g)-derivation of X. Then $d(x) = d(x * 0) = (d(x) * f(0)) \wedge (g(x) * d(0)) = (d(x) * 0) \wedge (g(x) * d(0)) = d(x) \wedge (g(x) * d(0)) = (g(x) * d(0)) * ((g(x) * d(0)) * d(x)) = (g(x) * d(0)) * ((g(x) * d(x)) * d(0))$ (using Lemma 1.11(3)) as X is medial. Combining Theorem 1.4 and Theorem 1.5 we have, $(g(x) * d(0)) * ((g(x) * d(x)) * d(0)) = g(x) * (g(x) * d(x)) = d(x) \wedge g(x)$.

Definition 2.6 *An (f, g)-derivation d of a G-algebra is said to be regular if $d(0) = 0$.*

Proposition 2.7 *Let d be a (r, l)-(f, g)-derivation of a G-algebra X. Then $d(x) = f(x) \wedge d(x)$, for all $x \in X$ if and only if d is regular.*

Proof: "$\Rightarrow$" Suppose that $d(x) = f(x) \wedge d(x)$. Then $d(0) = f(0) \wedge d(0) = 0 \wedge d(0) = 0$ and hence d is regular.
"$\Leftarrow$" Suppose that d is regular. Then $d(x) = d(x * 0) = (f(x) * d(0)) \wedge (d(x) * g(0)) = (f(x) * 0) \wedge (d(x) * 0) = f(x) \wedge d(x)$.

Theorem 2.8 *Let X be a G-algebra. If d is a (l, r)-(f, g)-derivation of X then $d(a + b) = d(a) + f(b)$, for all $a, b \in X$. Similarly, if d is a (r, l)-(f, g)-derivation of X then $d(a + b) = f(a) + d(b)$.*

Proof: We have $d(a+b) = d(a*(0*b)) = d(a)*f(0*b) = d(a)*(f(0)*f(b)) = d(a) * (0 * f(b)) = d(a) + f(b)$. The second part is proved similarly.

Corollary 2.9 *If G-algebra is 0-commutative then $d(a + b) = d(b + a)$.*

Proposition 2.10 *Let X be a G-algebra and d be an (f, g)-derivation of X. Then:*

(1) $d(0) \in L_p(X)$,

(2) *For all $a \in L_p(X)$, we have $f(a)$ and $g(a) \in L_p(X)$.*

Proof:

(1) From Proposition 1.3*(2)*, we know that if $x * d(0) = 0$ then $x = d(0)$ and so $d(0) \in L_p(X)$.

(2) Let $a \in L_p(X)$. Then $a = 0 * (0 * a)$. Therefore, $f(a) = f(0 * (0 * a)) = f(0) * f(0 * a) = 0 * (f(0) * f(a)) = 0 * (0 * f(a))$. Hence $f(a) \in L_p(X)$. Similarly it can be shown that $g(a) \in L_p(X)$.

Proposition 2.11 *Let X be a G-algebra and let d be a (l,r)-(f,g)-derivation of X. Then:*

(1) *For all $a \in L_p(X)$, $d(a) = d(0) + f(a)$,*

(2) *For all $a \in L_p(X)$, $d(a) \in L_p(X)$,*

(3) *For all $a, b \in L_p(X)$, $d(a + b) = d(a) + d(b) - d(0)$.*

Proof:

(1) Let $a \in L_p(X)$. Then $a = 0 * (0 * a)$. Therefore, $d(a) = d(0 * (0 * a)) = (d(0) * f(0 * a)) \wedge (g(0) * d(0 * a)) = d(0) * (f(0) * f(a)) = d(0) * (0 * f(a)) = d(0) + f(a)$.

(2) Let $a \in L_p(X)$. Using *(1)*, we have $d(a) = d(0) + f(a)$, hence $d(a) \in L_p(X)$ as $d(0)$ and $f(a) \in L_p(X)$ (from Proposition 2.10).

(3) Let $a, b \in L_p(X)$. Then, using *(1)*, $d(a + b) = d(0) + f(a + b) = d(0) + f(a) + f(b) = f(a) + d(b) + d(0) - d(0) = d(a) + d(b) - d(0)$.

Proposition 2.12 *Let X be a G-algebra and d be a (r,l)-(f,g)-derivation of X. Then:*

(1) *For all $a \in G(X)$, $d(a) \in G(X)$,*

(2) *For all $a \in L_p(X)$, $d(a) \in L_p(X)$,*

(3) *For all $a \in X$, $d(a) = f(a) + d(0)$,*

(4) *For all $a, b \in X$, $d(a + b) = d(a) + d(b) - d(0)$.*

Proof:

(1) Let $a \in G(X)$. Then $a = 0 * a$ and so $d(a) = d(0 * a) = f(0) * d(a) = 0 * d(a)$. Therefore $d(a) \in G(X)$.

(2) Let $a \in L_p(X)$. Then $a = 0 * (0 * a)$. Hence $d(a) = f(0) * d(0 * a) = 0 * (0 * d(a))$. Therefore, $d(a) \in L_p(X)$.

(3) Let $a \in X$. Then $d(a) = d(a * 0) = f(a) * d(0)$. As $d(0) \in G(X)$, then $d(a) = f(a) * (0 * d(0)) = f(a) + d(0)$.

(4) Similar prove to Proposition 2.11 *(3)*.

Proposition 2.13 *Let X be a G-algebra, d a (r,l)-(f,g)-derivation of X and f the identity map on X. Then d is the identity map on X if and only if d is regular.*

Proof: Let d be the identity map on X. From Proposition 2.12 *(3)*, $d(a) = d(0) + f(a)$, and so $a = d(0) + a$ which gives $d(0) = 0$. On the other hand, let $d(0) = 0$. Then $d(a) = d(a * 0) = f(a) * d(0) = a * 0 = a$.

Theorem 2.14 *Let X be a G-algebra and d an (f,g)-derivation of X. If there exists $a \in X$ such that, for all $x \in X$, either $d(x) * f(a) = 0$ or $f(x) * d(a) = 0$ then d is regular.*

Proof: Let $a \in X$ such that $d(x) * f(a) = 0$, for all $x \in X$. Then $0 = d(x) * f(a) = f(x) * d(a) = d(x * a)$. Hence $d(0) = 0$. This proves that d is regular.

Definition 2.15 *Let X be a G-algebra and d_1, d_2 be an (f,g)-derivation of X. We define $d_1 \circ d_2 : X \to X$ by $(d_1 \circ d_2)(x) = d_1(d_2(x))$, for all $x \in X$.*

Theorem 2.16 *Let d be a regular (f,g)-derivation of a G-algebra X. If $d^2(x) = 0$, for all $x \in L_p(X)$ then $(f \circ d)(x) = \frac{1}{2}((f \circ d)(0))$ for all $x \in L_p(X)$.*

Proof: Let $x \in L_p(X)$ such that $d^2(x) = 0$. Therefore $x+x \in L_p(X)$ and so $d^2(x+x) = 0$. So we have $0 = d^2(x+x) = d(d(x+x)) = d(0) + f(d(x+x)) = 0 + f(d(x)+d(x)-d(0)) = 2f(d(x)) - f(d(0))$ and so $(f \circ d)(x) = \frac{1}{2}((f \circ d)(0))$.

Corollary 2.17 *Let d_1 and d_2 be two regular (f,g)-derivations of a G-algebra X. If $(d_1 \circ d_2)(x) = 0$, for all $x \in L_p(X)$ then $(f \circ d_2)(x) = \frac{1}{2}((f \circ d_2)(0))$ for all $x \in L_p(X)$.*

Definition 2.18 *A G-algebra X is said to be torsion free if it satisfies $x + x = 0$ implies $x = 0$, for all $x \in X$. If there exists a nonzero element $x \in X$ such that $x + x = 0$ then X is not a torsion free.*

Example 2.19 *The algebra X given in Example 1.14, is not a torsion free as there exist a non zero element $1 \in X$ such that $1 + 1 = 1 * (0 * 1) = 0$.*

Theorem 2.20 *Let X be a torsion free G-algebra and d be a (l,r)-(f,g)-derivation of X such that $(f \circ d)(x) = d(x)$. Then for all $x \in L_p(X)$, if $d^2(x) = 0$ we have $d(x) = 0$.*

Proof: Let $x \in L_p(X)$ such that $d^2(x) = 0$. As $x \in L_p(X)$, we have $x + x \in L_p(X)$. Thus $0 = d^2(x+x) = d(d(x+x)) = d(0) + f(d(x+x)) = d(0) + d(x+x) = d(0) + d(x) + d(x) - d(0) = d(x) + d(x)$. As X is torsion free, then $d(x) + d(x) = 0$ implies $d(x) = 0$. This proves that d is the zero map.

Theorem 2.21 *Let X be a torsion free G-algebra, d_2 an (f,g)-derivation and d_1 a (l,r)-(f,g)-derivation of X such that for all $x \in X$, $(f \circ d_2)(x) = d_2(x)$. If $(d_1 \circ d_2)(x) = 0$, for all $x \in X$, then $d_2(x) = 0$, for all $x \in X$.*

Proof: Suppose that $(d_1 \circ d_2)(x) = 0$. Then $0 = (d_1 \circ d_2)(x+x) = d_1(d_2(x+x)) = d_1(0) + f(d_2(x+x)) = d_1(0) + d_2(x+x) = d_1(0) + (d_2(x) + d_2(x) - d_2(0)) = d_1(0) - d_2(0) + (d_2(x) + d_2(x)) = (d_1(0) * d_2(0)) + (d_2(x) + d_2(x)) = (d_1(0) * (0 * d_2(0)) + (d_2(x) + d_2(x)) = (d_1(0) + d_2(0)) + (d_2(x) + d_2(x)) = (d_1(0) + f(d_2(0))) + (d_2(x) + d_2(x)) = d_1(d_2(0)) + (d_2(x) + d_2(x)) = (d_1 \circ d_2)(0) + (d_2(x) + d_2(x))$. Having $(d_1 \circ d_2)(0) = 0$ gives $d_2(x) + d_2(x) = 0$ and so $d_2(x) = 0$ as X is torsion free.

Theorem 2.22 *Let X be a G-algebra. If d_1 and d_2 are (f,g)-derivations of X such that $f^2 = f$. Then $d_1 \circ d_2$ is an (f,g)-derivation of X.*

Proof: Let $x, y \in X$. Then $(d_1 \circ d_2)(x * y) = d_1(d_2(x * y)) = d_1((d_2(x) * f(y)) \wedge (g(x) * d_2(y))) = d_1(d_2(x) * f(y)) = (d_1(d_2(x)) * f(f(y))) = d_1(d_2(x)) * f(y) = (d_1 \circ d_2(x) * f(y)))$. Therefore, $d_1 \circ d_2$ is a (r,l)-(f,g)-derivation and it is proved similarly that $d_1 \circ d_2$ is a (l,r)-(f,g)-derivation of X.

References

[1] H. A. S. Abujabal, and N. O. Al-Shehri, Some results on derivations of BCI-algebras, *Journal of Natural Sciences and Mathematics-Lahore,* **46** (2006), no. 1-2, 13-19.

[2] D. Al-Kadi, f_q-Derivations of G-Algebra, *International Journal of Mathematics and Mathematical Sciences,* **2016** (2016), 1-5. https://doi.org/10.1155/2016/9276096

[3] D. Al-Kadi and R. Hosny, Notes on G-algebra and its derivations, *Mathematical Sciences Letters,* **6** (2017), no. 3, 287-292. https://doi.org/10.18576/msl/060310

[4] N. O. Al-Shehrie, Derivations of B-algebras, *Journal of King Abdulaziz University-Science*, **22** (2010), no. 1, 71-83. https://doi.org/10.4197/sci.22-1.5

[5] R. K. Bandru and N. Rafi, On G-algebras, *Scientia Magna,* **8** (2012), no. 3, 1-7.

[6] H. E. Bell and G. Mason, On derivations in near-rings and rings, *Math. J. Okayama Univ.,* **34** (1992), no. 1, 135-144.

[7] Y. B. Jun, and X. L. Xin, On Derivations of BCI-algebras, *Information Sciences,* **159** (2004), no. 3-4, 167-176.
https://doi.org/10.1016/j.ins.2003.03.001

[8] K. J. Lee, A new kind of derivations in BCI-algebras, *Applied Mathematical Sciences,* **7** (2013), no. 84, 4185-4194.
https://doi.org/10.12988/ams.2013.24298

[9] G. Muhiuddin and A. M. Al-Roqi, On (α, β)-derivations in BCI-algebras, *Discrete Dynamics in Nature and Society,* **2012** (2012), 1-11.
https://doi.org/10.1155/2012/403209

[10] G. Muhiuddin and A. M. Al-Roqi, On t-derivations of BCI-algebras, *Abstract and Applied Analysis,* **2012** (2012), 1-12.
https://doi.org/10.1155/2012/872784

[11] G. Muhiuddin and A. M. Al-Roqi, Generalizations of Derivations in BCI-Algebras, *Appl. Math. and Information Sciences,* **9** (2015), no. 1, 89-94.
https://doi.org/10.12785/amis/010113

[12] C. Prabpayak and U. Leerawat, On derivations of BCC-algebras, *Kasetsart J. (Nat. Sci.),* **43** (2009), 398-401.

Received: September 17, 2017; Published: November 6, 2017