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Groups with Isomorphic Tables of Marks Orders: 32, 48, 72 and 80[1]

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Abstract

We use GAP [2] to prove that groups of orders 32, 48, 72 and 80 with isomorphic tables of marks must be isomorphic groups. This continues previous work done for groups of order less than 96, and leaves only the case of order 64, to be dealt with separately in another paper, because it has hundreds of isomorphism classes of groups.

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1 Introduction

The table of marks of a finite group provides a considerable amount of information about it, but it is generally not enough to determine the isomorphism

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class of the group. In [1] you can find the basic definitions and properties of isomorphisms of tables of marks. In this paper we shall use mostly the fact that groups with isomorphic tables of marks must have the same order, the same number of elements of a given order, the same number of normal subgroups of a given order, and the same number of conjugacy classes of subgroups of a given order. We also rely heavily on the fact that abelian groups are completely determined by their tables of marks, that is, if G is an abelian group and H is an arbitrary group whose table of marks is isomorphic to the table of marks of G, then G and H are isomorphic groups. Therefore, in this paper we only have to deal with non-abelian groups.

In [3] you can find two non-isomorphic groups of order 96 whose tables of marks are isomorphic. This paper is one in a series of papers aimed at proving that those groups of order 96 are the smallest possible counterexamples. In [4] all cases are covered except orders 32, 48, 64, 72 and 80. Here we use the computer algebra package GAP (Groups, Algorithms and Programming) to prove that any two groups of orders 32,48, 72 or 80 with isomorphic tables of marks must be isomorphic as groups. The case of order 64 will be dealt with in one final paper, since there are hundreds of isomorphism classes of groups of that order.

2 Notations and conventions.

Throughout this paper we shall use the following notations:

Every family of groups (for example, the family of groups of order 32, groups of order 48, etc) will be ordered, giving each of its groups an id that completely identifies it. For example, the 20th group of order 32 is the quaternion group $Q(32)$.

CA = Number of elements of a group of a given order. After this, we usually write : followed by a list of the ids of the groups in the family with that particular invariant. For example, inside the family of groups of order 32, when we write CA=[[1, 1], [2, 11], [4, 4], [8, 16]]: 7 it means that of all groups of order 32, the only group which has 1 element of order 1, 11 elements of order 2, 4 elements of order 4 and 16 elements of order 8, is the group with id=7. In particular, this implies that this group is completely determined by its table of marks, since an isomorphism of tables of marks must preserve the order of the group and the orders of the elements.

NSN(n) = Number of normal subgroups of the n-th group of the family. This invariant must also be preserved by an isomorphism of tables of marks.

CCS(n) = Conjugacy classes of subgroups of the n-th group of the family. This invariant must also be preserved by an isomorphism of tables of marks.

Example: In the family of groups of order 32, we see that CA = [[1, 1], [2, 3], [4, 12], [8, 16]]: 4, 8, 12. This means that subgroups number 4, number 8 and number 12 could only have tables of marks isomorphic to one of those subgroups (number 4, 8 or 12). An extra computation: NSN(4) =18, NSN(8) =12, NSN(12) =16, yields that these three subgroups are completely determined by their tables of marks.

3 Non-abelian groups of order 32

2.$(C_4 \times C_2) : C4$
4. $C_8 : C_4$
5.$(C_8 \times C_2) : C_2$
6.$((C_4 \times C_2) : C_2$
7. $((C_8 : C_2) : C_2$
8. $C_2.((C_4 \times C_2) : C_2) = (C_2 \times C_2).(C_4 \times C_2)$
9. $(C_8 \times C_2) : C_2$
10. $Q_8 : C_4$
11. $(C_4 \times C_4) : C_2$
12. $C_4 : C_8$
13. $C_8 : C_4$
14. $C_8 : C_4$
15. $C_4.D_8 = C_4.(C_4 \times C_2)$
17. $C_{16} : C_2$
18. D_{32}
19. QD_{32}
20. Q_{32}
24. $(C_4 \times C_4) : C_2$
27. $(C_2 \times C_2 \times C_2 \times C_2) : C_2$
28. $(C_4 \times C_2 \times C_2) : C_2$
29. $(C_2 \times Q_8) : C_2$
30. $(C_4 \times C_2 \times C_2) : C_2$
31. $(C_4 \times C_4) : C_2$
32. $(C_2 \times C_2).(C_2 \times C_2 \times C_2)$
33. $(C_4 \times C_4) : C_2$
34. $(C_4 \times C_4) : C_2$
35. $C_4 : Q_8$
38. $(C_8 \times C_2) : C_2$
42.$(C8 \times C2) : C2$
43.$(C2 \times D8) : C2$
44.$(C2 \times Q8) : C2$
49.$(C2 \times D8) : C2$
50.$(C2 \times Q8) : C2$

Groups of order 32 sorted according to the orders of their elements:

- CA = [[1, 1], [2, 11], [4, 20]]: 6.
- CA = [[1, 1], [2, 11], [4, 4], [8, 16]]: 7.
- CA = [[1, 1], [2, 3], [4, 4], [8, 24]]: 15.
- CA = [[1, 1], [2, 3], [4, 4], [8, 8], [16, 16]]: 17.
- CA = [[1, 1], [2, 17], [4, 2], [8, 4], [16, 8]]: 18.
- CA = [[1, 1], [2, 9], [4, 10], [8, 4], [16, 8]]: 19.
- CA = [[1, 1], [2, 1], [4, 18], [8, 4], [16, 8]]: 20.
- CA = [[1, 1], [2, 1], [4, 18], [8, 4], [16, 8]]: 28.
- CA = [[1, 1], [2, 15], [4, 8], [8, 8]]: 43.

- CA=[[1, 1], [2, 11], [4, 4], [8, 16]]: 7.
- CA=[[1, 1], [2, 3], [4, 4], [8, 24]]: 15.
- CA=[[1, 1], [2, 3], [4, 4], [8, 8], [16, 16]]: 17.
- CA=[[1, 1], [2, 17], [4, 2], [8, 4], [16, 8]]: 18.
- CA=[[1, 1], [2, 9], [4, 10], [8, 4], [16, 8]]: 19.
- CA=[[1, 1], [2, 1], [4, 18], [8, 4], [16, 8]]: 20.
- CA=[[1, 1], [2, 15], [4, 16]]: 28.
- CA=[[1, 1], [2, 15], [4, 8], [8, 8]]: 43.
- CA = [[1, 1], [2, 7], [4, 24]]: 2, 24, 29, 33.
  NSN(2) = 26, NSN(24) =30, NSN(29) =24, NSN(33) =20.
- CA = [[1, 1], [2, 3], [4, 12], [8, 16]]: 4, 8, 12.
  NSN(4) =18, NSN(8) =12, NSN(12) =16.
- CA = [[1, 1], [2, 7], [4, 8], [8, 16]]: 5, 38.
  NSN(5) =16, NSN(38) =28.
- CA = [[1, 1], [2, 11], [4, 12], [8, 8]]: 9, 42.
  NSN(9) =14, NSN(42) =20

* CA = [[1, 1], [2, 3], [4, 20], [8, 8]]: 10, 13, 14.
  CCS(13) = [[1, 14], [2, 2], [4, 2]] = CCS(14).
  CCS(10) = [[1, 14], [2, 6], [4, 1]]. REMARK: The groups number 13 and 14 must be dealt with separately. You can find their tables of marks and the reason why they are non-isomorphic tables of marks (even though they are almost identical) at the end of this section.

- CA = [[1, 1], [2, 7], [4, 16], [8, 8]]: 11, 44.
  NSN(11) = 12, NSN(44) = 20.
- CA = [[1, 1], [2, 19], [4, 12]]: 27, 34, 49.
  NSN(49) =68.
  NSN(27) = 26 = NSN(34).
  CCS(27) = [[1, 26], [2, 38], [4, 1]],
  CCS(34) = [[1, 26], [2, 24], [4, 4]].

- CA = [[1, 1], [2, 11], [4, 20]]: 30, 31, 50.
  NSN(50) =68.
  NSN(30) = 22 = NSN(31).
  CCS(30) = [[1, 22], [2, 16], [4, 1]],
  CCS(31) = [[1, 22], [2, 14], [4, 2]].

- CA = [[1, 1], [2, 3], [4, 28]]: 32, 35.
  NSN(32) = 22, NSN(35) =26.

3.1 Table of marks of group 13 of order 32.

```
1:       32
   2:      16 16
   3:      16  .  16
   4:      16 .    .  16
   5:      8 8 8 8 8
   6:      8  8  .    .  .  8
   7:      8  8  .    .  .  .  8
   8:      8 .  .  8 .  .  .  2
   9:      8 .  8 .  .  .  .  .  2
   10:      4  4  4  4  4  4  4 .  .  4
   11:      4  4  4  4  4 .  .  .  2 .  2
   12:      4  4  4  4  4 .  .  2 .  .  .  2
   13:      4 4 .  ..4 .    .  .  .  .  .  4
   14:      4 4 .  .  .  4 .  .  .  .  .  .  .  4
   15:      2 2 2 2 2 2 2 2 .2 .    2 .  .2
   16:      2 2 2 2 2 2 2 .  .  2 .  .  2 2 .  2
   17:      2 2 2 2 2 2 2 .  2 2 2 .  .  .  .  .  2
   18:      1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

3.2 Table of marks of group 14 of order 32.

```
1:  32
   2:  16 16
   3:    16 .  16
   4:    16 .  .  16
   5:      8 8 8 8 8
   6:      8  8  .    .  .  8
   7:      8  8  .    .  .  .  8
   8:      8  .    8  .  .  .  .  2
   9:      8  .    8  .  .  .  .  .  2
   10:      4 4 4 4 4 4 4 .  .  4
   11:      4 4 4 4 4 .  .  .  2 .  2
```

```
12:      4 4 4 4 4 .  .  2 .  .  .  2
13:      4 4 .  .  .  4 .  .  .  .  .  .  4
14:      4 4 .  .  .  4 .  .  .  .  .  .  .  4
15:      2 2 2 2 2 2 2 2 .  2 .  2 .  .  2
16:      2 2 2 2 2 2 2 .  .  2 .  .  2 2 .  2
17:      2 2 2 2 2 2 2 .  2 2 2 .  .  .  .  .  2
18:      1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

These two groups have non-isomorphic tables of marks because the subgroups number 8 and 9 must correspond (either 8 goes to 8 or 8 goes to 9), but neither case is possible because one will have a zero where the other should have an 8 (in other words, the 8's are lined up in group 13, but they are out of sync for the group 14).

4 Non-abelian groups of order 48

1. $C_3 : C_{16}$
3. $(C_4 \times C_4) : C_3$
5. $C_{24} : C_2$
6. $C_{24} : C_2$
7. D_{48}
8. $C_3 : Q_{16}$
10. $(C_3 : C_8) : C_2$
12. $(C_3 : C_4) : C_4$
13. $C_{12} : C_4$
14. $(C_{12} \times C_2) : C_2$
15. $(C_3 \times D_8) : C_2$
16. $(C_3 : C_8) : C_2$
17. $(C_3 \times Q_8) : C_2$
18. $C_3 : Q_{16}$
19. $(C_2 \times (C_3 : C_4)) : C_2$
28. $C_2.S_4 = SL(2,3).C_2$
29. $GL(2,3)$
30. $A_4 : C_4$
33. $SL(2,3) : C_2$
37. $(C_{12} \times C_2) : C_2$
39. $(C_2 \times (C_3 : C_4)) : C_2$
41. $(C_4 \times S_3) : C_2$
50. $(C_2 \times C_2 \times C_2 \times C_2) : C_3$

Groups of order 48 sorted according to the orders of their elements:

- CA = [[1, 1], [2, 1], [3, 2], [4, 2], [6, 2], [8, 4], [12, 4], [16, 24], [24, 8]]: 1.
- CA = [[1, 1], [2, 3], [3, 32], [4, 12]]: 3.
- CA = [[1, 1], [2, 7], [3, 2], [4, 8], [6, 2], [8, 16], [12, 4], [24, 8]]: 5.
- CA = [[1, 1], [2, 13], [3, 2], [4, 14], [6, 2], [8, 4], [12, 4], [24, 8]]: 6.

- CA = [[1, 1], [2, 25], [3, 2], [4, 2], [6, 2], [8, 4], [12, 4], [24, 8]]: 7.
- CA = [[1, 1], [2, 1], [3, 2], [4, 26], [6, 2], [8, 4], [12, 4], [24, 8]]: 8.
- CA = [[1, 1], [2, 3], [3, 2], [4, 4], [6, 6], [8, 24], [12, 8]]: 10.
- CA = [[1, 1], [2, 17], [3, 2], [4, 2], [6, 10], [8, 12], [12, 4]]: 15.
- CA = [[1, 1], [2, 5], [3, 2], [4, 14], [6, 10], [8, 12], [12, 4]]: 16.
- CA = [[1, 1], [2, 13], [3, 2], [4, 6], [6, 2], [8, 12], [12, 12]]: 17.
- CA = [[1, 1], [2, 1], [3, 2], [4, 18], [6, 2], [8, 12], [12, 12]]: 18.
- CA = [[1, 1], [2, 7], [3, 2], [4, 24], [6, 14]]: 19.
- CA = [[1, 1], [2, 1], [3, 8], [4, 18], [6, 8], [8, 12]]: 28.
- CA = [[1, 1], [2, 13], [3, 8], [4, 6], [6, 8], [8, 12]]: 29.
- CA = [[1, 1], [2, 7], [3, 8], [4, 24], [6, 8]]: 30.
- CA = [[1, 1], [2, 7], [3, 8], [4, 8], [6, 8], [12, 16]]: 33.
- CA = [[1, 1], [2, 11], [3, 2], [4, 20], [6, 10], [12, 4]]: 39.
- CA = [[1, 1], [2, 19], [3, 2], [4, 12], [6, 2], [12, 12]]: 41.
- CA = [[1, 1], [2, 15], [3, 32]]: 50.
- CA = [[1, 1], [2, 3], [3, 2], [4, 28], [6, 6], [12, 8]]: 12, 13.
  NSN(12) = 17, NSN(13) =19.
- CA = [[1, 1], [2, 15], [3, 2], [4, 16], [6, 6], [12, 8]]: 14, 37.
  NSN(14) = 17, NSN(37) =23.

5 Non-abelian groups of order 72

1. $C_9 : C_8$	22. $(C_6 \times S_3) : C_2$
3. $Q_8 : C_9$	23. $(C_6 \times S_3) : C_2$
4. $C_9 : Q_8$	24. $(C_3 \times C_3) : Q_8$
6. D_{72}	31. $(C_3 \times C_3) : Q_8$
8. $(C_{18} \times C_2) : C_2$	33. $(C_{12} \times C_3) : C_2$
13. $(C_3 \times C_3) : C_8$	35. $(C_6 \times C_6) : C_2$
15. $((C_2 \times C_2) : C_9) : C_2$	40. $(S_3 \times S_3) : C_2$
19. $(C_3 \times C_3) : C_8$	41. $(C_3 \times C_3) : Q_8$
21. $(C_3 \times (C_3 : C_4)) : C_2$	43. $(C_3 \times A_4) : C_2$

Groups of order 72 sorted according to the orders of their elements:

- CA= [[1, 1], [2, 1], [3, 2], [4, 2], [6, 2], [8, 36], [9, 6],[12, 4],[18,6], [36, 12]]=1.
- CA= [[1, 1], [2, 1], [3, 2], [4, 6], [6, 2], [9, 24], [12, 12],[18, 24]]=3.
- CA= [[1, 1], [2, 1], [3, 2], [4, 38], [6, 2], [9, 6], [12, 4],[18, 6],[36, 12]]=4.
- CA= [[1, 1], [2, 37], [3, 2], [4, 2], [6, 2], [9, 6], [12, 4],[18, 6],[36, 12]]=6.
- CA= [[1, 1], [2, 21], [3, 2], [4, 18], [6, 6], [9, 6], [18, 18]]=8.
- CA= [[1, 1], [2, 1], [3, 8], [4, 2], [6, 8], [8, 36], [12, 16]]=13.
- CA= [[1, 1], [2, 21], [3, 2], [4, 18], [6, 6], [9, 24]]=15.
- CA= [[1, 1], [2, 1], [3, 8], [4, 18], [6, 8], [8, 36]]=19.
- CA= [[1, 1], [2, 19], [3, 8], [4, 12], [6, 8], [12, 24]]=21.
- CA= [[1, 1], [2, 13], [3, 8], [4, 18], [6, 32]]=22.
- CA= [[1, 1], [2, 25], [3, 8], [4, 6], [6, 20], [12, 12]]=23.
- CA= [[1, 1], [2, 1], [3, 8], [4, 30], [6, 8], [12, 24]]=24.
- CA= [[1, 1], [2, 1], [3, 8], [4, 38], [6, 8], [12, 16]]=31.
- CA= [[1, 1], [2, 37], [3, 8], [4, 2], [6, 8], [12, 16]]=33.
- CA= [[1, 1], [2, 9], [3, 8], [4, 18], [8, 36]]=39.

- CA= [[1, 1], [2, 9], [3, 8], [4, 54]]= 41.
- CA= [[1, 1], [2, 21], [3, 26], [4, 18], [6, 6]]= 43.
- CA= [[1, 1], [2, 21], [3, 8], [4, 18], [6, 24]]= 35,40. NSN(35)=21,NSN(40)= 7.

6 Non-abelian groups of order 80

1. $C_5 : C_{16}$
3. $C_5 : C_{16}$
5. $C_{40} : C_2$
6. $C_{40} : C_2$
7. D_{80}
8. $C_5 : Q_{16}$
10. $(C_5 : C_8) : C_2$
12. $(C_5 : C_4) : C_4$
13. $C_{20} : C_4$
14. $(C_{20} \times C_2) : C_2$
15. $(C_5 \times D_8) : C_2$
16. $(C_5 : C_8) : C_2$
17. $(C_5 \times Q_8) : C_2$
18. $C_5 : Q_{16}$
19. $(C_2 \times (C_5 : C_4)) : C_2$
28. $(C_5 : C_8) : C_2$
29. $(C_5 : C_8) : C_2$
31. $C_{20} : C_4$
33. $(C_5 : C_8) : C_2$
34. $(C_2 \times (C_5 : C_4)) : C_2$
38. $(C_{20} \times C_2) : C_2$
40. $(C_2 \times (C_5 : C_4)) : C_2$
42. $(C_4 \times D_{10}) : C_2$
49. $(C_2 \times C_2 \times C_2 \times C_2) : C_5$

Groups of order 80 sorted according to the orders of their elements:

- CA= [[1, 1], [2, 1], [4, 2], [5, 4], [8, 4], [10, 4], [16, 40],[20, 8], [40, 16] = 1.
- CA= [[1, 1], [2, 1], [4, 2], [5, 4], [8, 20], [10, 4], [16, 40], [20, 8]]= 3.
- CA= [[1, 1], [2, 11], [4, 12], [5, 4], [8, 24], [10, 4], [20, 8], [40, 16]]= 5.
- CA= [[1, 1], [2, 21], [4, 22], [5, 4], [8, 4], [10, 4], [20, 8], [40, 16]]= 6.
- CA= [[1, 1], [2, 41], [4, 2], [5, 4], [8, 4], [10, 4], [20, 8], [40, 16]]= 7.
- CA= [[1, 1], [2, 1], [4, 42], [5, 4], [8, 4], [10, 4], [20, 8], [40, 16]]= 8.
- CA= [[1, 1], [2, 3], [4, 4], [5, 4], [8, 40], [10, 12], [20, 16]]= 10.

- CA = [[1, 1], [2, 25], [4, 2], [5, 4], [8, 20], [10, 20], [20, 8]]= 15.
- CA= [[1, 1], [2, 5], [4, 22], [5, 4], [8, 20], [10, 20], [20, 8]]=16.
- CA= [[1, 1], [2, 21], [4, 6], [5, 4], [8, 20], [10, 4], [20, 24]]= 17.
- CA= [[1, 1], [2, 1], [4, 26], [5, 4], [8, 20], [10, 4], [20, 24]]= 18.
- CA= [[1, 1], [2, 7], [4, 40], [5, 4], [10, 28]]= 19.
- CA= [[1, 1], [2, 11], [4, 52], [5, 4], [10, 4], [20, 8]]= 31.
- CA = [[1, 1], [2, 3], [4, 20], [5, 4], [8, 40], [10, 12]]= 33.
- CA= [[1, 1], [2, 23], [4, 40], [5, 4], [10, 12]] = 34.
- CA= [[1, 1], [2, 15], [4, 32], [5, 4], [10, 20], [20, 8]]= 40.
- CA= [[1, 1], [2, 31], [4, 16], [5, 4], [10, 4], [20, 24]]= 42.
- CA= [[1, 1], [2, 15], [5, 64]]= 49.
- CA= [[1, 1], [2, 3], [4, 44], [5, 4], [10, 12], [20, 16]]= 12, 13.
  NSN(12) = 17, NSN(13) = 19.
- CA= [[1, 1], [2, 23], [4, 24], [5, 4], [10, 12], [20, 16]]= 14,38.
  NSN(14) = 17, NSN(38) = 23.
- CA= [[1, 1], [2, 11], [4, 12], [5, 4], [8, 40], [10, 4], [20, 8]]= 28, 29.
  NSN(28) = 14, NSN(29)= 12.

7 Conclusions

These computations prove that any group of order 32, 48, 72 or 80 is completely determined by its table of marks. It only remains to prove this for groups of order 64 in order to establish that the smallest example of non-isomorphic groups with isomorphic tables of marks must be of order 96.

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