Two Pentagonal Number Primality Tests and Twin Prime Counting in Arithmetic Progressions of Modulus 24

Werner Hürlimann
Swiss Mathematical Society, University of Fribourg
CH-1700 Fribourg, Switzerland

Abstract

Two generalized pentagonal number based primality tests for numbers in the arithmetic progressions $24n \pm 1$ are obtained. Their application suggest a new Diophantine approach to the existence of an infinite number of twin primes of the form $(24n - 1, 24n + 1)$.

Mathematics Subject Classification: 11A51, 11B25, 11D85, 11P32

Keywords: primality; compositeness; pentagonal number; arithmetic progression; quadratic Diophantine curve; divisor function; twin prime conjecture

1. Introduction

According to Riesel [13], Chapter 4, one distinguishes between primality tests and compositeness tests. Given a number $N$, a successful primality test on it proves that $N$ is prime, and a successful compositeness test proves that $N$ is composite. A stringent primality test states a condition on $N$, which implies that $N$ is prime if it is fulfilled and $N$ is composite otherwise. Usually, primality tests are often quite complicated (e.g. the Rabin-Miller test or the AKS test by Agrawal et al. [1], as presented in Schoof [14]) or only applicable to numbers of a special form. For a brief history before the computer age consult Mollin [12].

The considered pentagonal number based primality tests apply only to numbers that belong to the two arithmetic progressions $24n \pm 1$. They exploit a relationship
between numbers from these sequences and generalised pentagonal numbers of the form \( \frac{1}{2}m(3m \pm 1) \) (sequence A001318 in the OEIS of Sloane [15]). In Section 2, we consider the infinite matrix \( S = (S_{k,j}) \) of pentagonal S-numbers defined by \( S_{k,2i-1} = f^-(k,i), \quad S_{k,2i} = f^+(k,i), \quad k = 1,2,\ldots, \quad i = 1,2,\ldots, \) with

\[
f^\pm(x,y) = (x-1)(6y \pm 1) + \frac{1}{2} y(3y \pm 1), \quad x, y = 1,2,\ldots, \tag{2.1}
\]

a binary function of degree two. We prove that \( 24n+1 \) is prime if, and only if, the number \( n \) is not an S-number. Moreover, composite numbers \( 24n+1 \) are always generated by S-numbers \( n \). Section 3 considers a similar infinite partially truncated matrix \( T = (T_{k,j}) \) of pentagonal T-numbers and derives a primality test for numbers in the arithmetic progression \( 24n-1 \). A Diophantine application to the twin prime conjecture follows in Section 4.

## 2. Primality test for numbers in the arithmetic progression \( 24n+1 \)

Starting point are the binary functions of degree two defined by

\[
f^\pm(x,y) = (x-1)(6y \pm 1) + \frac{1}{2} y(3y \pm 1), \quad x, y = 1,2,\ldots, \tag{2.1}
\]

which are affine transforms of generalised pentagonal numbers \( \frac{1}{2} y(3y \pm 1) \) and numbers of the form \( 6y \pm 1 \). Consider the infinite matrix of natural numbers \( S = (S_{k,j}), \quad k = 1,2,\ldots, \quad j = 1,2,\ldots, \) called pentagonal S-numbers, and defined by

\[
S_{k,2i-1} = f^-(k,i), \quad S_{k,2i} = f^+(k,i), \quad k = 1,2,\ldots, \quad i = 1,2,\ldots. \tag{2.2}
\]

We claim that S-numbers of the form \( S_{k,j} = n \) for some \( (k,j) \) always generate composite numbers in the arithmetic progression \( 24n+1 \), and that natural numbers \( n \), which cannot be represented as \( S_{k,j} = n \), necessarily lead to prime numbers of the form \( 24n+1 \). The first assertion is almost trivial in view of the identity

\[
24f^\pm(x,y) + 1 = (6y \pm 1) \cdot (6y \pm 1 + 24(x-1)), \quad x, y = 1,2,\ldots. \tag{2.3}
\]

The second assertion is less elementary, but not very difficult to show. What is remarkable is the fact that the stated conditions characterize the totality of prime and composite numbers in this special arithmetic progression.

**Theorem 2.1 (Primality test with pentagonal S-numbers).** A number of the form \( 24n+1, \quad n = 1,2,\ldots, \) is prime if, and only if, \( n \) is not a pentagonal S-number.
Proof. By definition (2.2), it suffices to show that the two quadratic Diophantine curves \( f^\pm(x, y) = n \) have positive integral solutions \( x, y \geq 1 \) if, and only if, the number \( N = 24n + 1 \) is composite. To solve these two Diophantine equations we follow Krätzel [10], Section 6.1. We distinguish between two cases.

Case 1: \( f^+(x, y) = n \)

This equation is of the form

\[
ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{2.4}
\]

with coefficients \( a = 0, b = 6, c = \frac{3}{2}, d = 1, e = -\frac{11}{2}, f = -(1 + n) \). Since \( 4ac - b^2 = -36 \) is negative, the curve (2.4) is a hyperbola. Multiply (2.4) with \( 4c(4ac - b^2) = -6 \cdot 36 \) and consider the transformation of variables

\[
x' = (4ac - b^2)x + 2cd - be = -36(x - 1), \quad y' = bx + 2cy + e = 6x + 3y - \frac{11}{2},
\]

which implies that (2.4) is equivalent with the equation \( 36y'^2 - x'^2 = 9N, N = 24n + 1 \). Setting further \( x' = \pm 3X, y' = \pm \frac{1}{2}Y \), one obtains the equation \( Y^2 - X^2 = N \).

Case 2: \( f^-(x, y) = n \)

The curve (2.4) with \( a = 0, b = 6, c = \frac{3}{2}, d = -1, e = -\frac{11}{2}, f = -(n - 1) \) is also a hyperbola. The transformation of variables \( x' = \pm 3X, y' = \pm \frac{1}{2}Y \), with \( x' = -36(x - 1), y' = 6x + 3y - \frac{11}{2} \) leads to the same equation \( Y^2 - X^2 = N \).

Let now \( N = 24n + 1 = p \) be a prime. In both cases, set \( Y - X = t, Y + X = d \). One has to solve the equation \( t \cdot d = p \), hence \( (t, d) = (1, p) \) or \( (t, d) = (p, 1) \). It follows that \( x' = \pm 3X = \pm 3 \frac{p - 1}{2} = \pm 36n, y' = \pm \frac{1}{2}Y = \pm \frac{p - 1}{4} = \pm (6n + \frac{1}{2}) \). To make \( x > 0, y \geq 0 \) choose \( x' = -36n, y' = 6n + \frac{1}{2} \). Then, transforming back, one obtains \( x = n + 1, y = 0 \) in Case 1 and \( y = \frac{1}{2} \) in Case 2, which shows that \( n \) is not of the form \( S_{k, 2d} \) nor of the form \( S_{k, 2i - 1} \) in (2.2). Therefore, if \( N = 24n + 1 = p \) is a prime, then \( n \) is not an S-number. It remains to show that if \( N = 24n + 1 \) is composite, then \( n \) is an S-number. First of all, one observes that \( N \) is not divisible by 2 and 3. Therefore, this number contains a factor of the form \( t = 6i \pm 1 \leq \sqrt{N} \) for some \( i = 1, 2, ..., \). The cofactor \( d \) such that \( t \cdot d = N \) satisfies the inequalities \( d \geq \sqrt{N} \geq t \), hence \( d = 6i \pm 1 + z \) for some natural number \( z \geq 0 \). It follows that
\[ N = td = (6i \pm 1)(6i \pm 1 + z) = (6i \pm 1)^2 + (6i \pm 1)z = 24\frac{1}{2}i(3i \pm 1) + 1 + (6i \pm 1)z. \]

To be of the form \( N = 24n + 1 \) one must have \( z = 24(k - 1) \) for some \( k = 1, 2, \ldots \). It follows that \( N = 24n + 1 = (6i \pm 1)(6i \pm 1 + 24(k - 1)) \) and \( n \) is an S-number by the identity (2.3).

**Remarks 2.1.** The study of squares in arithmetic progressions is a prominent topic, which goes back to Diophantus, who constructed three squares in arithmetic progression (see Dickson [5], Chapter XIV). Fermat stated in 1640 that there are no four squares in arithmetic progression, which has been proved by Euler [6], Lebesgue [11] and others (more recent proofs by van der Poorten [16] and Conrad [3]). With the above, one sees that \( N = 24n + 1 \) is a square if, and only if, \( x = 1 \) in (2.3), that is \( n = \frac{1}{2} y(3y \pm 1), y = 1, 2, \ldots \), belongs to the sequence of generalized pentagonal numbers, and necessarily \( N = (6y \pm 1)^2 \), a result stated in Section 1 of González-Jiménez and Xarles [7]. This sequence plays a primordial role in the strong Rudin conjecture, which has been partially proved by these authors. An earlier discussion of Rudin’s conjecture is Bombieri et al. [2].

### 3. Primality test for numbers in the arithmetic progression \( 24n-1 \)

Given are the binary functions of degree two

\[ g^\pm(x, y) = (x - 1)(6y \pm 1) - \frac{1}{2} y(3y \pm 1), \quad x, y = 1, 2, \ldots, \tag{3.1} \]

which are also transforms of \( \frac{1}{2} y(3y \pm 1) \) and \( 6y \pm 1 \). Consider the infinite truncated matrix of natural numbers \( T = (T_{k, j}), k = 2, 3, \ldots, j = 1, 2, \ldots, 4k - 5 \), called *pentagonal T-numbers*, which are defined by

\[
T_{k, 2i-1} = g^-(k, i), \quad k = 2, 3, \ldots, i = 1, 2, \ldots, 2k - 2 \\
T_{k, 2i} = g^+(k, i), \quad k = 2, 3, \ldots, i = 1, 2, \ldots, 2k - 3 . \tag{3.2}
\]

The truncation is motivated as follows. As in Section 1, we would like that T-numbers of the form \( T_{k, j} = n \) for some \( (k, j) \) always generate composite numbers in the arithmetic progression \( 24n-1 \), and that natural numbers \( n \), which cannot be represented as \( T_{k, j} = n \), necessarily lead to prime numbers of the form \( 24n-1 \). Similarly to (2.3) one has the identity

\[ 24g^\pm(x, y) - 1 = (6y \pm 1) - (24x - 1)(6y \pm 1)), \quad x, y = 1, 2, \ldots. \tag{3.3} \]
If \( 6y \pm 1 \) is prime this expression will generate composite numbers only if the second factor exceeds one, that is \( y \leq 4x - 5 \). Now, if this is satisfied, one easily sees that \( T \)-numbers defined by (3.2) are strictly positive and strictly increasing such that the inequalities \( T_{k,2i} > T_{k,2i-1} > T_{k,2(i-1)} \) hold. We show the following complete characterization of the totality of prime and composite numbers in the arithmetic progression \( 24n - 1 \).

**Theorem 3.1** (Primality test with pentagonal \( T \)-numbers). A number of the form \( 24n - 1, n = 1, 2, \ldots, \) is prime if, and only if, \( n \) is not a pentagonal \( T \)-number.

**Proof.** We claim that the Diophantine curves \( g^+(x, y) = n \) have integral solutions \( x = 2, 3, \ldots, y = 1, 2, \ldots, 2(x-1) \) if, and only if \( N = 24n - 1 \) is composite.

**Case 1:** \( g^+(x, y) = n \)

With \( a = 0, b = 6, c = -\frac{3}{2}, d = 1, e = -\frac{13}{2}, f = -(1 + n) \), this equation is of the form

\[
ax^2 + bxy + cy^2 + dx + ey + f = 0.
\] (3.4)

Since \( 4ac - b^2 = -36 \) is negative, the curve (3.4) is a hyperbola. Multiply (3.4) with \( 4c(4ac - b^2) = 6 \cdot 36 \) and consider the transformation of variables

\[
x' = (4ac - b^2)x + 2cd - be = -36(x-1), \quad y' = bx + 2cy + e = 6x - 3y - \frac{13}{2},
\]

which implies that (3.4) is equivalent with the equation \( x'^2 - 36y'^2 = 9N, N = 24n - 1 \). Setting further \( x' = \pm 3X, y' = \pm \frac{1}{2}Y \), one obtains the equation \( X^2 - Y^2 = N \).

**Case 2:** \( g^-(x, y) = n \)

The hyperbola (3.4) with \( a = 0, b = 6, c = -\frac{3}{2}, d = -1, e = -\frac{11}{2}, f = -(n-1) \) is transformed to the equation \( X^2 - Y^2 = N \) setting \( x' = \pm 3X, y' = \pm \frac{1}{2}Y \), with \( x' = -36(x-1), y' = 6x - 3y - \frac{11}{2} \).

If \( N = 24n - 1 = p \) is a prime, then set in both cases \( X - Y = t, X + Y = d \). Solving the equation \( t \cdot d = p \), one gets \( (t, d) = (1, p) \) or \( (t, d) = (p, 1) \). It follows that \( x' = \pm 3X = \pm 3 \frac{p+1}{2} = \pm 36n, y' = \pm \frac{1}{2}Y = \pm \frac{p+1}{2} = \pm (6n - \frac{1}{2}) \). To make \( x > 0, y \geq 0 \) choose \( x' = -36n \), \( y' = 6n - \frac{1}{2} \). Transforming back, one has \( x = n + 1, y = 0 \) in Case 1 and \( y = \frac{1}{3} \) in Case 2, hence \( n \) is not of the form \( T_{k,2i} \) nor of the form \( T_{k,2i-1} \) in (3.2). Therefore, if \( N = 24n - 1 = p \) is a prime, then \( n \) is not
an T-number. It remains to show that if \( N = 24n - 1 \) is composite, then \( n \) is an T-number. Since \( N \) is not divisible by 2 and 3, it contains a prime factor \( t = 6i \pm 1 \) for some \( i = 1, 2, \ldots \). The cofactor \( d \) such that \( t \cdot d = N \) can be written as \( d = z - (6i \pm 1) \) for some natural number \( z > (6i \pm 1) \). It follows that

\[
N = td = (6i \pm 1)(z - (6i \pm 1)) = (6i \pm 1)z - (6i \pm 1)^2 = (6i \pm 1)z - 24\frac{1}{2}i(3i \pm 1) - 1.
\]

To be of the form \( N = 24n - 1 \) one must have \( z = 24(k-1) \) for some \( k = 2, 3, \ldots \). It follows that \( N = 24n - 1 = (6i \pm 1)(24(k-1) - (6i \pm 1)) \). The second factor must be non-trivial, hence \( i \leq 4k - 5 \) and \( n \) is an T-number by the identity (3.3). ∨

4. Counting S- and T-numbers: application to twin primes

We state some counting formulas for S- and T-numbers and apply them to determine the number of twin primes \( (24n - 1, 24n + 1) \) in finite sets \( \{1, 2, \ldots, N\} \). The following notations are used:

- \( S_c(N) \): The set of S-numbers in \( \{1, 2, \ldots, N\} \) such that \( 24n + 1 \) is composite
- \( T_c(N) \): The set of T-numbers in \( \{1, 2, \ldots, N\} \) such that \( 24n - 1 \) is composite
- \( S_p(N) = \overline{S_c(N)} \): The set of \( n \in \{1, 2, \ldots, N\} \) such that \( 24n + 1 \) is prime
- \( T_p(N) = \overline{T_c(N)} \): The set of \( n \in \{1, 2, \ldots, N\} \) such that \( 24n - 1 \) is prime

The intersection \( S_p(N) \cap T_p(N) \) consists of those \( n \in \{1, 2, \ldots, N\} \) such that \( (24n - 1, 24n + 1) \) is a twin prime. With \( |M| \) the cardinality of the set \( M \), one obtains the counting formula

\[
|S_p(N) \cap T_p(N)| = |S_c(N) \cap T_c(N)| = |S_c(N)| + |T_c(N)| - |S_c(N) \cup T_c(N)|
= |S_p(N)| + |T_p(N)| - |S_c(N) \cap T_c(N)| \quad (4.1)
= |S_p(N)| + |T_p(N)| + |S_c(N) \cap T_c(N)| - N.
\]

**Theorem 4.1** (Twin prime conjecture in arithmetic progressions of modulus 24)

There exists an infinity of twin primes \( (24n - 1, 24n + 1) \) if, and only if, the following inequality holds:

\[
|S_p(N)| + |T_p(N)| + |S_c(N) \cap T_c(N)| > N, \text{ for all } N \geq 3. \quad (4.2)
\]
In fact, the identity (4.1) and Theorem 4.1 hold for arbitrary arithmetic progressions \( qn \pm 1 \), \( n \in \{1,2,\ldots,N\} \), if one identifies the sets \( S_e(N) \), \( T_e(N) \) as the sets of composite numbers in \( qn \pm 1 \), and \( S_p(N) \), \( T_p(N) \) as the sets of primes in \( qn \pm 1 \). What is special to the modulus \( q = 24 \) is the Diophantine interpretation. The set \( S_e(N) \cap T_e(N) \) represents the numbers \( n \in \{1,2,\ldots,N\} \) that are simultaneously \( S \)- and \( T \)-numbers. From the proofs of Theorem 2.1 and 3.1 one sees that \( S_p(N) \cap T_p(N) \) coincides with the numbers \( n \in \{1,2,\ldots,N\} \) such that the intersection of the two hyperbolas

\[
Y^2 - X^2 = 24n + 1, \quad U^2 - V^2 = 24n - 1, \quad (4.3)
\]

have integral solutions \((X,Y,U,V)\) that satisfy the conditions

\[
x = 1 + \frac{n}{2} \in \{1,2,\ldots\}, \quad y = \frac{1}{6}(Y \pm 1 - X) \in \{1,2,\ldots\}, \\
u = 1 + \frac{n}{2} \in \{1,2,\ldots\}, \quad v = \frac{1}{6}(U \pm 1 - V) \in \{1,2,\ldots,2(u-1)\}. \quad (4.4)
\]

Alternatively, and this holds for arbitrary arithmetic progressions \( qn \pm 1 \), the cardinality of the set \( S_e(N) \cap T_e(N) \) is determined by the following formula (as usual \( d(n) \) denotes the divisor function, and \( 1\{\cdot\} \) is the indicator function)

\[
|S_e(N) \cap T_e(N)| = \sum_{n=1}^{N} [1\{d(qn-1) > 2\} \cdot 1\{d(qn+1) > 2\}]. \quad (4.5)
\]

Table 4.1 illustrates Theorem 4.1 for a small sample of computed values (to evaluate (4.5) use the sequence A000005 in Sloane’s OEIS).

**Table 4.1:** Number of twin primes in selected intervals

| \( N \) | \( |S_p(N)| \) | \( |T_p(N)| \) | \( |S_e(N) \cap T_e(N)| \) | \( |S_p(N) \cap T_p(N)| \) |
|---|---|---|---|---|
| 100 | 37 | 44 | 33 | 14 |
| 500 | 164 | 182 | 206 | 52 |
| 1000 | 315 | 333 | 436 | 84 |
| 2000 | 591 | 623 | 940 | 154 |
| 3000 | 871 | 900 | 1450 | 221 |
| 4000 | 1143 | 1152 | 1990 | 285 |

Together with the characterization Theorems 3.1 and 4.1 the definition of the \( S \)- and \( T \)-numbers in (2.2) and (3.2) can be used for the algorithmic generation of twin primes \((24n-1,24n+1)\) below a limit \( n \leq N \). Applying a sieve, it suffices to eliminate all \( S \)- and \( T \)-numbers below \( n \leq N \). The remaining \( n \in \{1,2,\ldots,N\} \)
yield the primes of the form $24n + 1$ respectively $24n - 1$, and those common values of $n \in \{1, 2, ..., N\}$ yield the twin primes. Table 4.2 illustrates.

Table 4.2: Twin primes $(24n - 1, 24n + 1)$ below $n \leq N = 1000$ (first prime listed)

<table>
<thead>
<tr>
<th>$n$</th>
<th>2087</th>
<th>4127</th>
<th>6959</th>
<th>10271</th>
<th>15647</th>
<th>20231</th>
</tr>
</thead>
<tbody>
<tr>
<td>71</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>191</td>
<td>2111</td>
<td>4271</td>
<td>7127</td>
<td>11159</td>
<td>15887</td>
<td>20639</td>
</tr>
<tr>
<td>239</td>
<td>2591</td>
<td>4799</td>
<td>7487</td>
<td>11351</td>
<td>16631</td>
<td>20807</td>
</tr>
<tr>
<td>311</td>
<td>2687</td>
<td>4967</td>
<td>7559</td>
<td>11831</td>
<td>17207</td>
<td>21191</td>
</tr>
<tr>
<td>431</td>
<td>2711</td>
<td>5231</td>
<td>8087</td>
<td>12071</td>
<td>18047</td>
<td>21599</td>
</tr>
<tr>
<td>599</td>
<td>2999</td>
<td>5279</td>
<td>8231</td>
<td>12239</td>
<td>19119</td>
<td>21647</td>
</tr>
<tr>
<td>1031</td>
<td>3119</td>
<td>5519</td>
<td>8999</td>
<td>13007</td>
<td>18287</td>
<td>21839</td>
</tr>
<tr>
<td>1151</td>
<td>3167</td>
<td>5639</td>
<td>9239</td>
<td>13679</td>
<td>18311</td>
<td>22271</td>
</tr>
<tr>
<td>1319</td>
<td>3359</td>
<td>5879</td>
<td>9431</td>
<td>14447</td>
<td>18911</td>
<td>22367</td>
</tr>
<tr>
<td>1487</td>
<td>3527</td>
<td>6359</td>
<td>9719</td>
<td>14591</td>
<td>19079</td>
<td>23039</td>
</tr>
<tr>
<td>1607</td>
<td>3671</td>
<td>6551</td>
<td>9767</td>
<td>15287</td>
<td>19751</td>
<td>23687</td>
</tr>
<tr>
<td>1871</td>
<td>3767</td>
<td>6791</td>
<td>10007</td>
<td>15359</td>
<td>19991</td>
<td>23831</td>
</tr>
</tbody>
</table>

Finally, the defined pentagonal S- and T-numbers suggest two new strategies to prove the twin prime conjecture. By Dirichlet’s theorem on the number of primes in arithmetic progressions, the asymptotic behaviour for the first two terms in (4.1) is known (e.g. Riesel [13], formula (2.40)). Therefore, one must further determine the asymptotic behaviour of the numbers $n \in \{1, 2, ..., N\}$ satisfying the Diophantine conditions (4.3)-(4.4) when $N \to \infty$ or obtain a sufficiently high lower bound for it. Equivalently, one must find an asymptotic formula or a lower bound that count the number of distinct S- and T-numbers. To achieve this seems difficult, and goes beyond the present study. However, readers specialized in the derivation of asymptotic formulas might appreciate these new possibilities.

Remark 4.1. Alternatively, it might be interesting to consider the sets that count the number of different representations of the equations $f^+(x, y) = n$ and $g^+(x, y) = n$. These sets can be viewed as generalized sets of S- and T-numbers that are denoted by $S^\text{multi}_c(N)$ respectively $T^\text{multi}_c(N)$. They count each S- or T-number according to its multiplicity taking into account the number of different solutions to the stated equations. It is not difficult to derive the following counting formulas for them (as usual $\lfloor \cdot \rfloor$ denotes the floor function)

$$|S^\text{multi}_c(N)| = \sum_{n=1}^{N} \left\lfloor \frac{d(24n+1)+1}{2} \right\rfloor - 1,$$

$$|T^\text{multi}_c(N)| = \sum_{n=1}^{N} \left\lfloor \frac{d(24n-1)+1}{2} \right\rfloor - 1.$$

Table 4.3 illustrates computation.
Table 4.3: Counting S- and T-numbers without and with multiplicity

| $N$  | $|S_c(N)|$ | $|S_c^{\text{mult}}(N)|$ | $|T_c(N)|$ | $|T_c^{\text{mult}}(N)|$ |
|------|-----------|----------------|---------|-----------------|
| 100  | 63        | 83             | 56      | 77              |
| 500  | 336       | 520            | 318     | 506             |
| 1000 | 685       | 1145           | 667     | 1124            |
| 2000 | 1409      | 2509           | 1377    | 2477            |
| 3000 | 2129      | 3955           | 2100    | 3912            |
| 4000 | 2857      | 5455           | 2848    | 5409            |

To conclude, let us mention that the method of the present note has been applied in Hürlimann [8] to obtain two triangular number based primality tests for numbers in the arithmetic progressions $8n \pm 1$. A similar application to the twin prime conjecture has also been given. Finally, we like to point out that different elementary twin prime characterization theorems have been obtained by Dilcher and Stolarsky [4], as well as Königsberg [9].

References


[8] W. Hürlimann, Two triangular number primality tests and twin prime counting


Received: May 13, 2016; Published: June 16, 2016