Application of Reduced Minimum Modulus and Minimum Modulus in an m-Isometry

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Abstract

In this paper we study the relationship among \((m, \infty) - isometry\), minimum modulus and reduced minimum modulus. We also find that there is a similar relationship among \((m, p) - isometry\), minimum modulus and reduced minimum modulus.

Keywords: \((m, p)\)-isometry, \((m, \infty)\)-isometry, reduced minimum modulus, minimum modulus, bounded below
1 Introduction

A bounded linear operator $T$ on a complex Hilbert space $H$ is called an $m$-isometry if it satisfies

$$
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{sk}T^k = 0
$$

These operators were introduced by Agler and were thoroughly studied by Agler and Stankus in a series of three paper [1],[2],[3].

Bayart introduced the definition of an $(m, \infty)$-isometric operator on a Banach space.

**Definition 1.1.** [4] Let $X$ be a Banach space, $T \in L(X)$, $m$ is a positive integer and $p \geq 1$ is real number, we say that $T$ is an $(m, p)$-isometry, if for any $x \in X$,

$$
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k(x)\|^p = 0
$$

Philipp Hoffmann, Michael Mackey and Micheál Ó Searcoid extended the definition of $(m, \infty)$-isometry, to include $p = \infty$.

**Definition 1.2.** [5] Let $X$ be a Banach space, $T$ is a linear operator, $m$ is a positive integer, we say that $T$ is an $(m, \infty)$-isometry, if for any $x \in X$,

$$
\max_{k=0, \ldots, m} \|T^k(x)\| = \max_{\text{k even}} \max_{\text{k odd}} \|T^k(x)\|
$$

In this paper, first, we introduce the properties of an $(m, \infty)$-isometry. Secondary, we study the relationship among $(m, \infty)$-isometry, the minimum modulus and the reduced minimum modulus[6]. Finally, we get some results of an $(m, p)$-isometry, which like the results of an $(m, \infty)$-isometry.

2 Properties of $(m, p)$-isometry

Philipp Hoffmann, Michael Mackey and Micheál Ó Searcoid has introduced some properties of an $(m, \infty)$-isometry.

**Proposition 2.1.** [5] Let $T$ be an $(m, \infty)$-isometry. Then there exists a norm on $X$, equivalent to $\|\cdot\|$, under which $T$ is an isometry.
Proposition 2.2. [5] Let \( T \) be an \((m, \infty)\) – isometry. Then, for all \( n \in \mathbb{N} \) and all \( x \in X \),

\[
\|T^n(x)\| \leq \max_{k=0, \ldots, m} \|T^k(x)\|
\]

In particular, \( T \) is power bounded by \( C = \max_{k=0, \ldots, m} \|T^k\| \).

From this theorem, when \( n = 1 \), we know \((m, \infty)\) – isometry is bounded linear operator.

Proposition 2.3. [5] Let \( T \) is \((m, \infty)\) – isometry, then \( T \) is bounded below.

Now, we give a new property of an \((m, \infty)\) – isometry.

Proposition 2.4. Let \( T \) is \((m, \infty)\) – isometry and \( \sigma_{ap}(T) \) is the approximate point spectrum, then \( \theta \notin \sigma_{ap}(T) \).

Proof. If \( \theta \in \sigma_{ap}(T) \), there exists a \( \{x_n\} \in X \) and \( \|x_n\| = 1 \), such that \( \lim_{n \to \infty} \|T(x_n)\| = 0 \). From Proposition 2.1, we have that there exists a norm \( \|\cdot\|_* \), such that \( m\|x\| \leq \|x\|_* \leq M\|x\| (M > m > 0) \) and \( \|T(x)\|_* = \|x\|_* \). Then we get that \( m\|T(x_n)\| \leq \|T(x_n)\|_* = \|x_n\|_* \leq M\|T(x_n)\| \). And because \( \lim_{n \to \infty} \|T(x_n)\| = 0 \), we have that \( \lim_{n \to \infty} \|x_n\|_* = 0 \). But for \( m\|x_n\| \leq \|x_n\|_* \leq M\|x_n\| \), we deduce that \( \lim_{n \to \infty} \|x_n\| = 0 \). This contradict with \( \|x_n\| = 1 \), so \( \theta \notin \sigma_{ap}(T) \). \( \square \)

3 The minimum modulus and the reduced minimum modulus of an \((m, \infty)\) – isometry

We give the definition of The reduced minimum modulus and the minimum modulus.

Definition 3.1. [6] Let \( X \) be a Banach space, \( T \in L(X) \) and \( x \in X \). we call \( \mu(T) \) is the minimum modulus of \( T \), if

\[
\mu(T) = \inf_{\|x\| = 1} \|T(x)\|
\]

Definition 3.2. [6] Let \( X \) be a Banach space, \( T \in L(X) \) and \( x \in X \). we call \( \gamma(T) \) is the reduced minimum modulus of \( T \), if

\[
\gamma(T) = \begin{cases} \inf\{\|T(x)\||d(x, N(T)) = 1\} & \text{if } T \neq 0 \\ \infty & \text{if } T = 0 \end{cases}
\]
In the following, we study the relationship between \((m, \infty)\)\textit{-isometry} and minimum modulus.

**Proposition 3.3.** \([6]\) Let \(T \in L(X)\). If \(\mu(T) > 0\), then \(T\) is bounded below.

Now, we need two properties of an \((m, \infty)\)\textit{-isometry} in \([5]\). Later, we give the relation between \((m, \infty)\)\textit{-isometry} and minimum modulus.

**Theorem 3.4.** \([5]\) Let \(T\) is an \((m, \infty)\)\textit{-isometry}, then \(T\) is injective.

**Theorem 3.5.** \([5]\) Let \(T\) is an \((m, \infty)\)\textit{-isometry} and \(R(T)\) is the range of \(T\). Then \(R(T)\) is closed.

**Theorem 3.6.** Let \(T\) is an \((m, \infty)\)\textit{-isometry}. Then \(\mu(T) > 0\).

*Proof.* If \(\mu(T) = 0\), we have \(\inf_{\|x\|=1} \|T(x)\| = 0\). Then there exist \(\{x_n\}\) and \(\|x_n\| = 1\), such that \(\lim_{n \to \infty} \|T(x_n)\| = 0\), so \(\lim_{n \to \infty} T(x_n) = 0\). Since \(T\) is an \((m, \infty)\)\textit{-isometry}, so \(T\) is bounded below. From proposition 3.5, \(R(T)\) is closed. As \(R(T)\) is a linear subspace, thus \(R(T)\) is complete. And because \(X\) is Banach space, \(T\) is open operator. In addition, \(T\) is an \((m, \infty)\)\textit{-isometry}, so \(T\) is bounded below and then \(T\) is injective, Hence \(T^{-1}\) is linear continuous operator. Thus we have

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} T^{-1}(T(x_n)) = T^{-1}(\lim_{n \to \infty} T(x_n)) = T^{-1}(\theta) = \theta
\]

This contradict with \(\|x_n\| = 1\). It means that \(\mu(T) \neq 0\), But \(\mu(T) \geq 0\), so \(\mu(T) > 0\). \(\square\)

As the following, we study the relationship between \((m, \infty)\)\textit{-isometry} and the reduced minimum modulus.

Let \(X\) be a Banach space, \(T \in L(X)\) and \(ker(T)\) is the kernel of \(T\). For any \([x] \in X/ker(T)\), we define \(\phi([x]) = T(x)\). Obviously, \(\phi\) is a map.

**Proposition 3.7.** \(\phi\) was as described above, then

1. \(\phi\) is a linear operator.
2. \(\phi\) is injective.
3. \(\phi\) is continuous.

*Proof.* (1)For any \([x_1], [x_2] \in X/ker(T)\),

\[
\phi([x_1] + [x_2]) = \phi([x_1 + x_2]) = T(x_1 + x_2) = T(x_1) + T(x_2) = \phi([x_1]) + \phi([x_2])
\]

\[
\phi(\alpha [x]) = \alpha T(x) = \alpha T(x) = \alpha [x]
\]

(2)For any \([x] \neq [\theta]\), we have that \(x \not\in ker(T)\). It is clearly that this implies that \(\phi[x] = T(x) \neq \theta\).
(3) Since $T$ is continuous, there exists $\rho > 0$, such that $\|T(x)\| \leq \rho \|x\|$. It means that $\|\phi[x]\| \leq \rho \|x\|$. We deduce that

$$\|\phi[x]\| \leq \inf_{x' \in [x]} \rho \|x'\| = \rho \inf_{x' \in [x]} \|x'\| = \rho \|[x]\|$$

So $\phi$ is continuous.

\[\square\]

**Remark 3.8.** Since $d(x, \ker(T)) = \inf_{x' \in \ker(T)} \|x - x'\| = \|[x]\|$, when $T \neq \theta$, we have that $\gamma(T) = \inf \{\|T(x)\|d(x, \ker(T)) = 1\} = \inf \{\|\phi[x]\|\|[x]\| = 1\}$. This means that $\gamma(T) = \mu(\phi)$.

**Theorem 3.9.** Let $X$ be a Banach space, $T \in L(X)(T \neq \theta)$ and $\mu(\phi)$ was as described above. If $\gamma(T) > 0$, then $\phi$ is bounded below.

**Proof.** For $\gamma(T) > 0$, we have that $\mu(\phi) > 0$. From proposition 3.3, we immediately get that $\phi$ is bounded below.

\[\square\]

**Theorem 3.10.** Let $T$ is an $(m, \infty)$–isometry, then $\gamma(T) > 0$.

**Proof.** If $\mu(\phi) = 0$, we have $\inf_{\|[x]\| = 1} \|\phi([x])\| = 0$. Then there exist $\{[x]_n\}$ and $\|[x]_n\| = 1$, such that $\lim_{n \to \infty} \|\phi([x]_n)\| = 0$, so $\lim_{n \to \infty} \phi([x]_n) = 0$. For $T$ is an $(m, \infty)$–isometry, so $R(T)$ is closed and then $R(\phi)$ is closed. And $R(\phi)$ is a linear subspace, thus $R(\phi)$ is complete. And because $X/\ker(T)$ is Banach space, $\phi$ is open operator. In addition, $\phi$ is injective, Hence $\phi^{-1}$ is linear continuous operator. Thus we have

$$\lim_{n \to \infty} [x]_n = \lim_{n \to \infty} \phi^{-1}(\phi([x]_n)) = \phi^{-1}(\lim_{n \to \infty} \phi([x]_n)) = \phi^{-1}(\theta) = \theta$$

This contradict with $\|[x]_n\| = 1$. It means that $\mu(\phi) \neq 0$. But $\mu(\phi) \geq 0$, so $\mu(\phi) > 0$. For $\gamma(T) = \mu(\phi)$, we have that $\gamma(T) > 0$.

\[\square\]

### 4 The minimum modulus and the reduced minimum modulus of an $(m,p)$–isometry

In this section, we study the results of an $(m,p)$–isometry, which were similary to the results of an $(m, \infty)$–isometry.

**Theorem 4.1.** Let $X$ be a Banach space, $T \in L(X)$ and $x \in X$, if $T$ is $m$-isometry and $\sigma_{ap}(T)$ is the approximate point spectrum, then $\theta \notin \sigma_{ap}(T)$.

**Proof.** If $\theta \in \sigma_{ap}(T)$, there exist $x_n$ and $\|x_n\| = 1$, such that $\lim_{n \to \infty} T(x_n) = \theta$.

And $T$ is continuous, we have that $\lim_{n \to \infty} T^k(x_n) = \theta(k = 1, 2, \cdots, m)$, then $\lim_{n \to \infty} \|T^k(x_n)\| = 0$. Thus $\|x_n\| = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k}(m-k)\|T^k(x_n)\|^p \to \theta(n \to \infty)$. This contradict with $\|x_n\| = 1$, so $\theta \notin \sigma_{ap}(T)$.

\[\square\]
As the proof of theorem 3.11, we have

**Theorem 4.2.** Let $X$ be a Banach space, $T \in L(X)$ and $x \in X$. $\gamma(T)$ is the reduce minimum modulus of $T$. If $T$ is an m-isometry, then $\gamma(T) > 0$.

**References**


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