Spinor Formulation of Sabban Frame of Curve on $S^2$

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Abstract

In this paper, we define spinor formulation of Sabban equations of curves on $S^2$. It is viewed that the Sabban equations for curves on $S^2$ can be reduced to a single equation for a vector with two complex components.

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1 Introduction

Given the importance of the triads of mutually orthogonal unit vectors in differential geometry, it is of interest that each such triad can be expressed in terms of a single vector with two complex components, called a spinor ([4], [6]). Spinors in general were discovered by Elie Cartan in 1913, [4]. Later, spinors were adopted by quantum mechanics in order to study the properties of the

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intrinsic angular momentum of the electron and other fermions. In mathematics, particularly in differential geometry and global analysis, spinors have since found broad applications to algebraic and differential topology, symplectic geometry, gauge theory and complex algebraic geometry. In the light of the existing studies, it was shown that the Frenet equations for curves in \( \mathbb{R}^3 \) can be reduced to a single equation for a vector with two complex components. It was found a consequence of the relationship between spinors and orthogonal triads of vectors. [5]. Ünal, Kisi and Tosun studied spinor Bishop equations of curves in \( \mathbb{E}^3 \). They investigated the spinor formulations of curves according to Bishop frames in \( \mathbb{E}^3 \). Also, the relations between spinor formulations of Bishop frames and Frenet frame are expressed, [1].

The aim of this paper is to show that the Sabban equations can be expressed with a single equation for a vector with two complex components. In Section 2, spinors and orthonormal bases are briefly explained; in Section 3 the spinor equivalent of the Sabban equations for a curve is obtained.

## 2 Spinors and Orthonormal Bases

The group of rotation about the origin denoted by \( \text{SO}(3) \) in \( \mathbb{R}^3 \) is homomorphic to the group of unitary complex \( 2 \times 2 \) matrices with the unit determinant denoted by \( \text{SU}(2) \). For this reason, there is a two-to-one homomorphism of \( \text{SU}(2) \) onto \( \text{SO}(3) \). The elements of \( \text{SU}(2) \) act on vectors with two complex components which are called spinors, whereas the elements of \( \text{SO}(3) \) act the vectors with three real components (see, [2], [8]). We can identify a spinor:

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]  

by means of three vectors \( a, b, c \in \mathbb{R}^3 \) such that

\[
a + ib = \psi^t \sigma \psi, \quad c = -\hat{\psi}^t \sigma \psi,
\]  

where \( \sigma \) is a vector whose Cartesian components are the complex symmetric \( 2 \times 2 \) matrices

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

The superscript \( t \) denotes transposition and \( \hat{\psi} \) is the mate of \( \psi \), \( \bar{\psi} \) be complex conjugation of \( \psi \), [12]. Therefore

\[
\hat{\psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = \begin{pmatrix} -\bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix}.
\]
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The vectors $a, b, c$ are explicitly given by

\[
\begin{align*}
a + ib &= (\psi_1^2 - \psi_2^2, i(\psi_1^2 + \psi_2^2), -2\psi_1\psi_2) \\
c &= (\psi\bar\psi_2 + \bar\psi_1\psi_2, i(\psi\bar\psi_2 - \bar\psi_1\psi_2), |\psi_1|^2 - |\psi_2|^2).
\end{align*}
\]

Therefore the vector $a + ib \in \mathbb{C}^3$ is an isotropic vector, by means of an explicit computation one finds that $a, b$ and $c$ are mutually orthogonal and $|a| = |b| = |c| = \bar{\psi}'\psi$, $\langle a \wedge b, c \rangle = \det(a, b, c) > 0$.

The other way around, given three mutually orthogonal vectors of the same magnitude, $a, b, c \in \mathbb{R}^3$ such that $\det(a, b, c) > 0$, then there exists a spinor, defined up to sign such that the equation (2.2) holds. Making use of the previous definitions for two arbitrary spinors $\phi$ and $\psi$, there exist the following equalities

\[
\begin{align*}
\bar{\phi}\sigma\psi &= -\hat{\phi}'\sigma\hat{\psi} \\
a\phi + b\psi &= \bar{a}\hat{\phi} + \bar{b}\hat{\psi} \\
\hat{\psi} &= -\psi,
\end{align*}
\]

where $a$ and $b$ are complex numbers ([4], [5]). The correspondence between the spinors and the orthogonal bases given by equation (2.2) is two-to-one because the spinors $\psi$ and $-\psi$ correspond to the same ordered orthogonal bases $\{a, b, c\}$ with $|a| = |b| = |c| = \bar{\psi}'\psi$, $\langle a \wedge b, c \rangle = \det(a, b, c) > 0$.

It is important to notice that the ordered triads $\{a, b, c\}$, $\{b, c, a\}$ and $\{c, b, a\}$ correspond to different spinors, [5]. Because the matrices $\sigma$ given by equation (2.3) are symmetric, any pairs of spinors $\phi$ and $\psi$ satisfy

\[
\phi'\sigma\phi = \psi'\sigma\psi.
\]

If $\psi$ is a spinor different from zero, the set $\{\hat{\psi}, \psi\}$ is linearly independent, ([4], [5]).

Let $\alpha : I \rightarrow E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in E^3$, with first and second curvature, $\kappa$ and $\tau$ respectively, the Frenet formulae is given by ([3], [7], [13])

\[
\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= -\tau N.
\end{align*}
\]

In the light of the foregoing findings, there exists a spinor, $\psi$, defined up to sign, such that

\[
N + iB = \psi'\sigma\psi, \quad T = -\hat{\psi}'\sigma\psi,
\]

\[\text{(2.8)}\]
with $\psi^2 = 1$. Since, the spinor $\psi$ represents the triad $\{N, B, T\}$ and the variations of this triad along the curve $\alpha$, given by the Frenet equations, must correspond to some expression for $\frac{d\psi}{ds}$.

**Theorem 2.1** If the two-component spinor $\psi$ represents the triad $\{N, B, T\}$ of a curve parametrized by its arclength $s$, according to (2.8), the Frenet equations are equivalent to the single spinor equation

$$\frac{d\psi}{ds} = \frac{1}{2}(-i\tau\psi + \kappa\hat{\psi}) \quad (2.9)$$

where $\tau$ and $\kappa$ denote the torsion and curvature of the curve, respectively. Equation (2.9) is called spinor Frenet equation, [5].

### 3 Spinor Formulation of Sabban Equations of Curves on $S^2$

The Euclidean 3-space $E^3$ be inner product given by

$$\langle x, y \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in E^3$. Recall that, the norm of an arbitrary vector $X \in E^3$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$. The curve $\alpha$ is called a unit speed curve if velocity vector $\alpha'$ of $\alpha$ satisfies $\|\alpha'\| = 1$. For vector $v, w \in E^3$, it said to be orthogonal if and only if $\langle v, w \rangle = 0$. The sphere of radius $r = 1$ and with center in the origin in the space $E^3$ is defined by

$$S^2 = \{P = (P_1, P_2, P_3) | \langle P, P \rangle = 1\}.$$  

Now, we give a new frame different from Frenet frame Let $\gamma : I \to S^2$ be a unit speed spherical curve. We denote $s$ as the arc-length parameter of $\gamma$. Let us denote by

$$\begin{cases}
\gamma(s) = \gamma(s) \\
t(s) = \gamma'(s) \\
d(s) = \gamma(s) \wedge t(s).
\end{cases} \quad (3.1)$$

We call $t(s)$ a unit tangent vector of $\gamma$. $\{\gamma, t, d\}$ frame is called the Sabban frame of $\gamma$ on $S^2$. Then we have the following spherical Frenet formulae of $\gamma$:

$$\begin{cases}
\gamma' = t \\
t' = -\gamma + \kappa_g d \\
d' = -\kappa_g t
\end{cases} \quad (3.2)$$

where is called the geodesic curvature of $\kappa_g$ on $S^2$ and

$$\kappa_g = \langle t', d \rangle, \quad ([10], [11]). \quad (3.3)$$
Theorem 3.1 If the two-component spinor $\eta$ represents the triad \( \{t, d, \gamma\} \) of a unit speed regular curve. Then, the Sabban frame is equivalent to the single spinor equation
\[
\frac{d\eta}{ds} = \frac{1}{2} \left( -i\kappa_g \eta + \hat{\eta} \right)
\]
where $\kappa_g$ is geodesic curvature on $S^2$.

Proof. Let $\eta$ be spinor;

\[
t + id = \eta t \sigma \eta, \quad \gamma = -\hat{\eta} t \sigma \eta,
\]
with $\eta t \eta = 1$, and the spinor represents the triad \( \{t, d, \gamma\} \). Additionally, the variations of this triad along the curve $\gamma$, given by the Sabban frame, must coincide with an expression for $\frac{d\eta}{ds}$. Differentiating the equation (3.5) with respect to $s$ and Sabban equations by given (3.2), we reach
\[
t' + id' = \frac{d}{ds}(\eta t \sigma \eta) = (\frac{d\eta}{ds})^t \sigma \eta + \eta^t \sigma (\frac{d\eta}{ds}).
\]
For $\{\eta, \hat{\eta}\}$ is a basis for the component spinors ($\eta \neq 0$), there are two functions as $f$ and $g$ such that
\[
\frac{d\eta}{ds} = f\eta + g\hat{\eta}
\]
where the functions $f$ and $g$ are possibly complex valued functions. Substituting equations (3.2) and (3.5) into (3.6), we find
\[
-\gamma + \kappa_g d + i(-\kappa_g t) = (\frac{d\eta}{ds})^t \sigma \eta + \eta^t \sigma (\frac{d\eta}{ds})
\]
\[
-\gamma - i\kappa_g (t + id) = (f\eta + g\hat{\eta})^t \sigma \eta + \eta^t \sigma (f\eta + g\hat{\eta})
\]
\[
-ik_g (t + id) - \gamma = 2f(t + id) - 2g(\gamma).
\]

From last equation, $f = -\frac{ik_g}{2}$, $g = \frac{1}{2}$. In this case
\[
\frac{d\eta}{ds} = \frac{1}{2} \left( -i\kappa_g \eta + \hat{\eta} \right)
\]

\[\blacksquare\]
References


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