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Chessboard Metric and Median Spaces

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Abstract

The median property is a geometric property satisfied by any three points on a metric space. In this article, we prove that a 2-dimensional vector space with its metric defined by the chessboard metric is not a median space.

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1 Introduction

A metric space X is called a *median space* if there exists an unique *median* among any three points in X. More precisely, For x, y in X, we define the *geodesic interval* $[x, y]$ by

$$[x, y] := \{t \in X \mid d_X(x, t) + d_X(t, y) = d_X(x, y)\}.$$

Definition 1.1 *In a metric space, for any three points a, b, c, a point x is called a median if $x \in [a, b] \cap [b, c] \cap [c, a]$.*

Definition 1.2 *A metric space is called a median space if for x, y, z in X, the intersection of geodesic intervals $[x, y], [y, z]$ and $[z, x]$ consists of one single point. That is, the median between any three points in X always exists and is unique.*

For more properties of median spaces, please see [2].

In this note, we look at a well-known metric on a plane and see if it is a median space. This article is of expository nature and is accessible to anybody who is interested in non-Euclidean geometry.

2 Chessboard metric

In the game of chess, the *chessboard distance* is defined by *the minimum number of moves needed by a king to go from one square to another on a chessboard.* A person will notice easily that it is the same as *Chebyshev distance*, or namely the L_∞−metric on a 2-dimensional vector space if he is familiar with the rules of a chess game. Mathematically speaking, for any two points $a = (x_1, y_1), b = (x_2, y_2)$ on a plane, the *chessboard metric* is defined by

$$D_{chess}(a, b) := \max(|x_1 - x_2|, |y_1 - y_2|).$$

We call a parallelogram *vertical* if the two parallel sides are vertical. Similarly, we call a parallelogram *horizontal* if the two parallel sides are horizontal. For any two points on a plane, we call the difference between the x-coordinates *the x-difference* and similarly, we call the difference between the y-coordinates *the y-difference*.

It is trivial that, if either the x-difference or the y-difference is zero, it means that the two points are at the same vertical or horizontal level, then there is an unique shortest path between them and hence $[a, b]$ consists of all points on this path. We consider the structure of $[a, b]$ in the case when both x-difference and y-difference are non-zero:

Lemma 2.1 *For any two points a, b on the plane associated with the chessboard metric, if both the x-difference and y-difference are non-zero, then the geodesic interval $[a, b]$ defined by the chessboard distance consists of all points in a parallelogram formed in the following manner:*

(1) If the x-difference is larger, then $[a, b]$ consists of all points in a horizontal parallelogram in which a and b are two vertices on the longer diagonal.

(2) If the y-difference is larger, then $[a, b]$ consists of all points in a vertical parallelogram in which a and b are two vertices on the longer diagonal.

(3) If both of them are equal, then $[a, b]$ consists of all points along the straight line from a and b.

Proof. We only consider any integral points a and b on the plane as the proof for non-integral points follows the same line of reasoning. Essentially we are looking at an infinitely large chessboard where a king always sits on the integral points and moves one step at a time. We should look at the shortest possible paths a king can move from one point to another. The geodesic interval $[a, b]$ is the union of all points along such paths from a to b.

Case (3) is trivial as the only moves a king can move is to take all the diagonal movements between the two points. So we only consider Case (1) and (2) from now on. Throughout the proof, we assume that both x-difference and y-difference are non-zero. To simplify the notations, let H be a horizontal move, V be a vertical move and D be a diagonal move by the king. Any path taken by a king can be written as a sequence of alphabets from $\{H, V, D\}$.

The proof can be done by induction on the (chessboard) distance between the two points. If the distance is 2, and let's say the x-difference is bigger (it basically means the x-difference is 2 and y-difference is 1), then HD and DH are the only 2 geodesic paths and hence it is a horizontal parallelogram. By induction, if the distance is n where x-difference is the larger one, then all the geodesic paths consist of n alphabets from $\{H, D\}$. More specifically, if the y-difference is k (where $k < n$), then each geodesic path contains k copies of D and $(n-k)$ copies of H. The total number of geodesic paths is equal to the number of ways to permute the k copies of D and $(n-k)$ copies of H in a set of n alphabets. Hence, on the plane, $[a, b]$ contains all the integral points bounded by the horizontal parallelogram formed by the two geodesic paths D...DH...H and H...HD...D (both of these paths have k copies of D and $(n-k)$ copies of H) from a to b. The same proof can be applied to the case where the y-difference is the bigger one by essentially replacing H by V in this proof. Done.

Definition 2.2 *In a metric space, for any three points a, b and c, the triple $\{a, b, c\}$ is called a triangle if none of these points is in the geodesic interval of the other two points. That is, $a \notin [b, c]$, $b \notin [a, c]$ and $c \notin [a, b]$.*

Example 2.3 *On a plane associated with the Euclidean metric, a triple $\{a, b, c\}$ is a triangle if a, b and c do not lie on the same straight line.*

Example 2.4 *On a plane associated with the chessboard metric, three points a, b and c do not form a triangle if one of the points is in the geodesic interval formed by the other two points. By Lemma 2.1, if $[a, b]$ forms a paralleogram (when the x-difference and y-difference are non-zero and not equal), then $\{a, b, c\}$ is a triangle if and only if c is not in the parallelogram of $[a.b]$.*

Lemma 2.5 *On a metric space, median always exist for any triple $\{a, b, c\}$ that is not a triangle.*

Proof. If a triple $\{a,b,c\}$ is not a triangle, then we can assume that $a \in [b,c]$. Since $a \in [a,b]$ and $a \in [c,a]$, it implies $a \in [a,b] \cap [b,c] \cap [c,a]$ and hence a is a median. Done.

The main theorem of this article is the following:

Theorem 2.6 *For a triangle $\{a,b,c\}$ on a plane associated with the chessboard metric, the intersection of geodesic intervals $[a,b],[b,c]$ and $[c,a]$ is an empty set. That is, median does not exist for any triangle on the plane associated with the chessboard metric.*

Proof. Let $a=(x_1,y_1), b=(x_2,y_2), c=(x_3,y_3)$ and denote the empty set by ϕ.

In the first case, if two points in $\{a,b,c\}$ are at the same horizontal level, say a and b where $x_1 < x_2$ and $y_1 = y_2$, then median has to lie on the horizontal geodesic path of $[a,b]$ and both $[b,c]$, $[c,a]$ must be horizontal parallelograms (which implies that $x_1 < x_3 < x_2$ by Lemma 2.1). But $[b,c] \cap [c,a] = \{c\}$ if $x_1 < x_3 < x_2$. Note that $c \notin [a,b]$ as $\{a,b,c\}$ is a triangle. Hence $[a,b] \cap [b,c] \cap [c,a] = \phi$ and there is no median. So, let's assume that $x_2 < x_3$. Then $[b,c] \cap [a,b] = \{b\}$. Note that $b \notin [a,c]$ as $\{a,b,c\}$ is a triangle. Hence, $[a,b] \cap [b,c] \cap [c,a] = \phi$ and there is no median. The case for $x_3 < x_1$ can be shown in the same manner.

Similarly, it can be shown that if two points in $\{a,b,c\}$ are at the same vertical level, then there is no median.

In the second case, a, b and c are at different horizontal levels such that $y_1 < y_2 < y_3$. We only look at the case where $x_1 < x_3$ (the case for $x_1 > x_3$ is essentially the same). If $x_1 < x_2 < x_3$, then $[a,b] \cap [b,c] = \{b\}$. Since $b \notin [a,c]$ (as $\{a,b,c\}$ is a triangle), $[a,b] \cap [b,c] \cap [c,a] = \phi$ and hence there is no median. So it implies that $x_2 \notin [x_1,x_3]$. We look at the case where $x_1 < x_3 < x_2$ (the case for $x_2 < x_1 < x_3$ is essentially the same). If either $[a,b]$ or $[b,c]$ is not a horizontal parallelogram. Then $[a,b] \cap [b,c] = \{b\}$ and there is no median as $b \notin [a,c]$ ($\{a,b,c\}$ is a triangle). It implies that $[a,b]$ and $[b,c]$ must be horizontal parallelograms such that $[a,b] \cap [b,c]$ consists of points that are on the same horizontal level as b. Note that all points in $[a,b] \cap [b,c]$ have x-coordinates in the interval (x_3,x_2). However, $[a,c]$ consists of all points with x-coordinates in (x_1,x_3). It implies that $[a,b] \cap [b,c] \cap [c,a] = \phi$ and hence median does not exist. Done.

Corollary 2.7 *The plane associated with the chessboard metric is not a median space.*

3 An example of median spaces

As a remark, another interesting metric on a plane is the so-called *taxicab metric* or *Manhattan metric*. The distance between two points is measured by the shortest path a taxi can travel between the two points on a grid. The latter name alludes to the grid layout of most streets on the Island of Manhattan in New York State. Mathematically, it is equivalent to the L_1-metric on the plane. For any two points $a = (x_1, y_1)$, $b = (x_2, y_2)$, the *taxicab metric* is defined by

$$D_{taxicab}(a, b) := |x_1 - x_2| + |y_1 - y_2|.$$

It is known that a plane associated with the taxicab metric is a median space. And it is surprisingly easy to find the median for any three points on the plane. For your reference, please take a look at [1]. It also contains many other interesting properties of the taxicab metric on the plane.

References

[1] E.F. Krause, *Taxicab Geometry*, Dover, 1987.

[2] E.R. Verheul, *Multimedians in Metric and Normed Spaces*, CWI Tract 91, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1993.

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