A Semi-Riemannian Manifold of Quasi-Constant Curvature Admits Lightlike Hypersurfaces

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Abstract

In this paper, we study the geometry of semi-Riemannian manifold of quasi-constant curvature admits lightlike hypersurfaces $M$. Our main result is two characterization theorems for such semi-Riemannian manifold such that $M$ is either screen homothetic or screen totally umbilical.

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1 Introduction and Preliminaries

In 1972, Chen-Yano [1] introduced the notion of Riemannian manifold of quasi-constant curvature as a Riemannian manifold $(\tilde{M}, \tilde{g})$ equipped with the curvature tensor $\tilde{R}$ satisfying the following condition:

\[ R(X, Y)Z = \alpha\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y\} \]
\[ + \beta\{\theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y \]
\[ + \tilde{g}(Y, Z)\theta(X)\zeta - \tilde{g}(X, Z)\theta(Y)\zeta\}, \]

where $\alpha$ and $\beta$ are smooth functions, $\zeta$ is a unit vector field on $\tilde{M}$ which is called the curvature vector field, and $\theta$ is a 1-form given by $\theta(X) = \tilde{g}(X, \zeta)$. In this case, if $\beta = 0$, then $\tilde{M}$ is a space of constant curvature.
In 1996, Duggal-Bejancu [2] published their work on the general theory of lightlike submanifolds. Since then there has been very active study on lightlike submanifolds. The theory of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds can be models of different types of horizons (event, Cauchy’s and Kruskal’s horizons). Although now we have lightlike version of a large variety of Riemannian submanifolds, the theory of semi-Riemannian manifold of quasi-constant curvature admits lightlike submanifolds is few known. There are some limited papers on particular subcases studied by Jin [5, 6] and Jin-Lee [7].

In this paper, we study the geometry of semi-Riemannian manifold $\overline{M}$ of quasi-constant curvature, equipped with the Levi-Civita connection $\overline{\nabla}$ and $\dim \overline{M} > 3$, admits special lightlike hypersurfaces. We prove the following two characterization theorems for such a semi-Riemannian manifold $\overline{M}$:

**Theorem 1.1.** Let $\overline{M}$ be a semi-Riemannian manifold of quasi-constant curvature. If $\overline{M}$ admits a screen homothetic lightlike hypersurface $M$ satisfying one of the following two conditions;

1. $\zeta$ is tangent to $M$, or
2. $\zeta$ is parallel with respect to $\overline{\nabla}$ and the transversal connection is flat,

then the functions $\alpha$ and $\beta$, defined by (1.1), vanish and $\overline{M}$ is flat.

**Theorem 1.2.** Let $\overline{M}$ be a semi-Riemannian manifold of quasi-constant curvature such that $\zeta$ is parallel with respect to $\overline{\nabla}$. If $\overline{M}$ admits a screen totally umbilical lightlike hypersurface $M$ satisfying one of the followings;

1. $\zeta$ is tangent to $M$, or
2. the transversal connection is flat,

then the functions $\alpha$ and $\beta$ vanish and $\overline{M}$ is flat.

It is known [2] that the normal bundle $TM^\perp$ of the lightlike hypersurfaces $(M, g)$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is a vector subbundle of the tangent bundle $TM$ and coincides with the radical distribution $\text{Rad}(TM)$. Therefore there exists a non-degenerate complementary vector bundle $S(TM)$ of $\text{Rad}(TM)$ in $TM$, which called a *screen distribution* on $M$, such that

$$TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM),$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on $M$, by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ and by $(\ast, \ast)_i$ the $i$-th equation of $(\ast, \ast)$. We use same notations for any others. For any null section $\xi$ of $\text{Rad}(TM)$, there exists a unique null section $N$ of a unique vector bundle $\text{tr}(TM)$ in $S(TM)^\perp$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)).$$
Then the tangent bundle $TM$ of $M$ is decomposed as follows;

$$TM = T\bar{M} \oplus tr(TM) = \{\text{Rad}(TM) \oplus tr(TM)\} \oplus_{\text{orth}} S(TM). \quad (1.3)$$

We call $tr(TM)$ and $N$ the transversal vector bundle and the null transversal vector field of $M$ with respect to the screen distribution $S(TM)$ respectively.

In the following, let $X, Y, Z$ and $W$ be the smooth vector fields on $M$, unless otherwise specified. Let $P$ be the projection morphism of $TM$ on $S(TM)$. Then the Gauss and Weingartan formulas for $M$ and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad (1.4)$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \quad (1.5)$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \quad (1.6)$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad (1.7)$$

where $\nabla$ and $\nabla^*$ are the linear connections on $TM$ and $S(TM)$ respectively, $B$ and $C$ are the local second fundamental forms on $TM$ and $S(TM)$ respectively, $A_N$ and $A^*_\xi$ are the shape operators and $\tau$ is a 1-form.

Since $\bar{\nabla}$ is torsion-free, $\nabla$ is also torsion-free and $B$ is symmetric. From the fact that $B(X,Y) = g(\bar{\nabla}_X Y, \xi)$, we know that $B$ is independent of the choice of a screen distribution $S(TM)$ and satisfies

$$B(X, \xi) = 0. \quad (1.8)$$

The induced connection $\nabla$ of $M$ is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \quad (1.9)$$

where $\eta$ is a 1-form such that

$$\eta(X) = \bar{g}(X, N).$$

But the connection $\nabla^*$ on $S(TM)$ is metric. The above two local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \quad (1.10)$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \quad (1.11)$$

## 2 The Ricci and scalar curvatures

Denote by $R$ and $R^*$ the curvature tensors of the induced connection $\nabla$ and $\nabla^*$ respectively. Using the Gauss-Weingarten equations for $M$ and $S(TM)$, we obtain the Gauss-Codazzi equations for $M$ and $S(TM)$ such that

$$\bar{g}(R(X,Y)Z, PW) = g(R(X,Y)Z, PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW), \quad (2.1)$$
The transversal connection $\nabla_{X,Y}$ for any $X, Y \in \Gamma(T\bar{M})$. Let $\dim \bar{M} = m + 2$. Locally, $\bar{Ric}$ is given by

$$\bar{Ric}(X, Y) = \sum_{i=1}^{m+2} \epsilon_i g(\bar{R}(E_i, X)Y, E_i),$$

where $\{E_1, \ldots, E_{m+2}\}$ is an orthonormal frame field of $\bar{M}$ and $\epsilon_i$ denotes the causal character of $E_i$. The scalar curvature $\bar{r}$ of $\bar{M}$ is defined by

$$\bar{r} = \sum_{i=1}^{m+2} \epsilon_i \bar{Ric}(E_i, E_i).$$

Consider a quasi-orthonormal frame field $\{\xi; W_a\}$ such that $Rad(TM) = \text{Span}\{\xi\}$ and $S(TM) = \text{Span}\{W_a\}$, and let $E = \{\xi, W_a, N\}$ be the corresponding frame field on $\bar{M}$. By using this frame field, we see that

$$\bar{Ric}(X, Y) = \sum_{a=1}^{m} \epsilon_a g(\bar{R}(W_a, X)Y, W_a)$$

$$+ g(\bar{R}(\xi, X)Y, N) + g(\bar{R}(N, X)Y, \xi),$$

$$\bar{r} = \bar{Ric}(\xi, \xi) + \bar{Ric}(N, N) + \sum_{a=1}^{m} \epsilon_a \bar{Ric}(W_a, W_a).$$

**Definition 1.** For any $X, Y \in \Gamma(TM)$, let $\nabla^\perp_X N = \pi(\nabla_X N)$, where $\pi$ is the projection morphism of $TM$ on the transversal vector bundle $tr(TM)$ of $M$. Then $\nabla^\perp$ is a linear connection on $tr(TM)$. We say that $\nabla^\perp$ is the transversal connection of $M$. We define the curvature tensor $R^\perp$ on $tr(TM)$ by

$$R^\perp(X, Y)N = \nabla^\perp_X \nabla^\perp_Y N - \nabla^\perp_Y \nabla^\perp_X N - \nabla^\perp_{[X,Y]} N.$$  

The transversal connection $\nabla^\perp$ is said to be flat [4] if $R^\perp$ vanishes identically.

**Theorem 2.1 [2, 4].** Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then the transversal connection is flat if and only if the 1-form $\tau$, defined by (1.5), is closed, i.e., $d\tau = 0$. In this case, we can take $\tau = 0$. 

\[ g(\tilde{R}(X,Y)Z, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \]

\[ + B(Y,Z)\tau(X) - B(X,Z)\tau(Y), \quad (2.2) \]

\[ g(\tilde{R}(X,Y)Z, N) = g(R(X,Y)Z, N), \quad (2.3) \]

\[ g(\tilde{R}(X,Y)\xi, N) = g(A_r^r X, A_r^r Y) - g(A_r^r Y, A_r^r X) - 2d\tau(X, Y), \quad (2.4) \]

\[ g(R(X,Y)PZ, PW) = g(R^*(X,Y)PZ, PW) \]

\[ + C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW), \quad (2.5) \]

\[ g(R(X,Y)PZ, N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X, PZ) + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \quad (2.6) \]
3 Proof of Theorem 1.1

Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. We shall assume that $\zeta$ is unit spacelike, without loss of generality. Let $f = \theta(\xi)$ and $e = \theta(N)$. Substituting (1.1) into (2.7), we have

$$\bar{Ric}(X,Y) = \{(m + 1)\alpha + \beta\}\bar{g}(X,Y) + m\beta \theta(X)\theta(Y). \quad (3.1)$$

Taking the scalar product with $N$ to (1.1), we have

$$\bar{g}(\bar{R}(X,Y)Z, N) = \{\alpha\eta(X) + e\beta\theta(X)\}g(Y, Z)$$

$$- \{\alpha\eta(Y) + e\beta\theta(Y)\}g(X, Z) + \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z). \quad (3.2)$$

**Theorem 3.1.** Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature admits a lightlike hypersurface. If $\beta = 0$, then $\alpha = 0$ and $\bar{M}$ is flat.

**Proof.** Substituting (3.1) into (2.8) and (2.10) by turns, we have

$$\bar{r} = (m + 1)\{(m + 2)\alpha + 2\beta\},$$

$$\bar{r} = m\{(m + 1)\alpha + \beta\} + m\beta(e^2 + f^2 + 1 - 2ef),$$

respectively. Comparing above two equations, we obtain

$$2(m + 1)\alpha + (m + 2)\beta = m\beta(e^2 + f^2 + 1 - 2ef). \quad (3.3)$$

If $\beta = 0$, then $(m + 1)\alpha = 0$. Thus $\alpha = 0$ and $\bar{M}$ is flat.

**Theorem 3.2.** Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature admits a lightlike hypersurface $M$. If the curvature vector field $\zeta$ of $\bar{M}$ is tangent to $M$, then the Ricci and scalar curvatures of $\bar{M}$ are given by

$$\bar{Ric} = \bar{r} \theta \otimes \theta, \quad \bar{r} = m\beta.$$

**Proof.** From (3.3) we deduce the following equation:

$$2\{(m + 1)\alpha + \beta\} = m\beta(e - f)^2.$$ 

Assume that $\zeta$ is tangent to $M$, i.e., $f = 0$. In this case, if $\zeta$ belongs to $\text{Rad}(TM)$, then we show that $\zeta = e\xi$. This implies

$$1 = \bar{g}(\zeta, \zeta) = e^2\bar{g}(\xi, \xi) = 0.$$ 

It is a contradiction. This enables one to choose a screen distribution $S(TM)$ which contains $\zeta$. This implies that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$ which we assume. Therefore $e = 0$. As $e = f = 0$, we show that

$$(m + 1)\alpha + \beta = 0. \quad (3.4)$$
Substituting (3.4) into (3.1), we obtain

\[ \bar{Ric}(X,Y) = m\beta \theta(X)\theta(Y), \quad \forall X, Y \in \Gamma(T\bar{M}). \]

Substituting this equation into (2.8) and using the fact \( \sum_{i=1}^{m+2} \epsilon_i \theta(E_i)\theta(E_i) = 1 \), we get \( \bar{r} = m\beta \). Thus we obtain our assertion.

**Definition 2.** A lightlike hypersurface \( M \) is screen homothetic [3] if

\[ A_N = \varphi A^*_\xi, \quad \text{or equivalently,} \quad C(X, PY) = \varphi B(X, Y), \quad (3.5) \]

where \( \varphi \) is a non-zero constant on \( M \). In particular, if \( \varphi = 0 \), i.e., \( C = A_N = 0 \), then \( M \) is called screen totally geodesic.

**Proof of Theorem 1.1.** Replacing \( Z \) by \( \xi \) to (3.2), we get

\[ \bar{g}(\bar{R}(X,Y)\xi, N) = f\beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}. \quad (3.6) \]

Comparing (2.4) and (3.6) and using the fact that \( A_N = \varphi A^*_\xi \), we obtain

\[ 2d\tau(X,Y) = f\beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}. \quad (3.7) \]

**Case (1).** In case \( \zeta \) is tangent to \( M \), i.e., \( f = 0 \): By Theorem 3.2, we show that \( \zeta \) belongs to \( S(TM) \). As \( f = 0 \), we have \( d\tau = 0 \) by (3.7). Therefore the transversal connection is flat by Theorem 2.1. Replacing \( X \) by \( \xi \) and \( Z \) by \( X \) to (3.2) and using the fact that \( f = e = 0 \), we have

\[ \bar{g}(\bar{R}(\xi,Y)X, N) = \alpha g(X,Y) + \beta \theta(X)\theta(Y). \quad (3.8) \]

As \( d\tau = 0 \), we can take \( \tau = 0 \) by Theorem 2.1. Replacing \( W \) by \( \xi \) to (1.1) and using (2.2) and the fact that \( \tau = f = 0 \), we have

\[ (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0. \quad (3.9) \]

Substituting (3.5) into (2.6) and using (3.9), we get \( \bar{g}(R(X,Y)PZ, N) = 0 \). From this equation, (2.3) and the fact that \( \bar{g}(\bar{R}(X,Y)\xi, N) = 0 \), we have

\[ \bar{g}(\bar{R}(X,Y)Z, N) = 0. \]

Replacing \( X \) by \( \xi \) and \( Z \) by \( X \) to this and then, comparing with (3.8), we have

\[ \beta \theta(X)\theta(Y) = -\alpha g(X,Y). \]

Taking \( X = Y = \zeta \) to this equation, we get \( \beta = -\alpha \). Substituting \( \beta = -\alpha \) into (3.4), we have \( \alpha = \beta = 0 \). Therefore, by (1.1), \( \bar{M} \) is flat.

**Case (2).** In case \( \zeta \) is parallel with respect to \( \bar{\nabla} \) and the transversal connection is flat: If \( \zeta \) is tangent to \( M \), then, by Case (1) we show that \( \alpha = \beta = 0 \).
and $\bar{M}$ is flat. Thus we may assume that $\zeta$ is not tangent to $M$, i.e., $f \neq 0$. As the transversal connection is flat, we get $d\tau = 0$. Therefore $\tau = 0$ and
\[
\beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\} = 0. \quad (3.10)
\]
Taking $X = PX$ and $Y = \xi$ to (3.10), we have $\beta\theta(PX) = 0$. If $\beta = 0$, then $\alpha = 0$ and $\bar{M}$ is flat. Thus we let $\beta \neq 0$. As $\theta(PX) = 0$, $\zeta$ is decomposed as
\[
\zeta = e\xi + fN. \quad (3.11)
\]
As $\bar{g}(\zeta,\zeta) = 1$, we get $2ef = 1$. Assume that $\zeta$ is parallel respect to $\bar{\nabla}$. Applying $\bar{\nabla}_X$ to (3.11) and using (1.4)$\sim$(1.8) and the fact that $A_\xi = \varphi A_\xi^\ast$, we show that $e$ and $f$ are constants and $(e + f\varphi)A_\xi^\ast X = 0$. As $(e + f\varphi)$ is a constant, either $e + f\varphi = 0$ or $A_\xi^\ast = 0$. Due to (3.5), the later case is equivalent to the condition: $M$ is both totally geodesic and screen totally geodesic.

In case $e + f\varphi = 0$: Replacing $W$ by $\xi$ to (1.1) and using (2.2), we have
\[
(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = f\beta\{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\}. \quad (3.12)
\]
Substituting (3.5) into (2.6) and using (2.3) and (3.12), we have
\[
\bar{g}(\bar{R}(X, Y)Z, N) = f\varphi\beta\{\theta(X)g(Y, Z) - \theta(Y)g(X, Z)\}, \quad (3.13)
\]
due to $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$. Replacing $X$ by $\xi$ to (3.13), we obtain
\[
\bar{g}(\bar{R}(\xi, Y)X, N) = f^2\varphi\beta g(X, Y).
\]
On the other hand, replacing $X$ by $\xi$ to (3.2) and using (3.10), we get
\[
\bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + ef\beta)g(X, Y).
\]
Comparing the last two equations, we have $\alpha + ef\beta = f^2\varphi\beta$, i.e., $\alpha = f(f\varphi - e)\beta$. Using this and the facts that $e + f\varphi = 0$ and $2ef = 1$, we get $\alpha = -\beta$. Substituting $\beta = -\alpha$ into (3.3), we have $\alpha(1 + e^2 + f^2) = 0$. This implies $\alpha = 0$. Therefore $\beta = 0$ and $\bar{M}$ is flat.

In case $M$ is screen totally geodesic: From (2.3) and (2.6), we get
\[
\bar{g}(\bar{R}(X, Y)PZ, N) = 0.
\]
From this and the fact that $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we show that
\[
\bar{g}(\bar{R}(X, Y)Z, N) = 0. \quad (3.14)
\]
Replacing $X$ by $\xi$ and $Z$ by $X$ to (3.2), we get
\[
\bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + ef\beta)g(X, Y). \quad (3.15)
\]
Replacing $X$ by $\xi$ and $Z$ by $X$ to (3.14) and then, comparing this result with (3.15), we have $\alpha + ef\beta = 0$. As $2ef = 1$, we get $\beta = -2\alpha$. Substituting $\beta = -2\alpha$ into (3.3), we have $\alpha\{1 - m(e^2 + f^2)\} = 0$. As $\{1 - m(e^2 + f^2)\}$ is a constant, if $\alpha \neq 0$, then we have $1 = m(e^2 + f^2)$. As $m > 1$, we get $e^2 + f^2 < m(e^2 + f^2) = 1$. Using this and the fact that $2ef = 1$, we have

$$(e - f)^2 = e^2 + f^2 - 2ef < 1 - 1 = 0.$$  

It is a contradiction. Thus $\alpha = 0$. Consequently $\beta = 0$ and $\bar{M}$ is flat.

By the procedure same as the method of Theorem 1.1 and by using $A_N = 0$ instead of $A_N = \varphi A_\xi$, we have the following corollary:

**Corollary 1.** Let $\bar{M}$ be a semi-Riemannian manifold of quasi-constant curvature such that $dim \bar{M} > 3$. If $\bar{M}$ admits a screen totally geodesic lightlike hypersurface $M$ satisfying one of the following two conditions;

1. $\zeta$ is tangent to $M$, or
2. $\zeta$ is parallel with respect to $\bar{\nabla}$ and the transversal connection is flat,

then the function $\alpha$ and $\beta$ vanish and $\bar{M}$ is flat.

4 Proof of Theorem 1.2

**Definition 3.** We say that $M$ is *screen totally umbilical* [2] if there exist a smooth function $\gamma$ such that $A_N X = \gamma PX$, or equivalently,

$$C(X, PY) = \gamma g(X, Y).$$  

(4.1)

Note that, in case $\gamma = 0$ on $\mathcal{U}$, $M$ is screen totally geodesic.

**Proof of Theorem 1.2.** Assume that $M$ is screen totally umbilical. From (2.4), (3.6) and the facts that $A_N X = \gamma PX$ and $A_\xi^*$ is self-adjoint, we get

$$2d\tau(X, Y) = f \beta\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}.$$  

Case (1). In case $\zeta$ is tangent to $M$, i.e., $f = 0$: By Theorem 3.2 and Case (1) of Theorem 1.1, we know that $\zeta$ belongs to $S(TM)$, $d\tau = 0$ and the transversal connection is flat. Replacing $Y$ by $\zeta$ to (1.4) and using (1.6) and the facts that $\zeta \in \Gamma(S(TM))$ and $\bar{\nabla}_X \zeta = 0$, we have

$$\nabla_X^* \zeta + C(X, \zeta) \zeta + B(X, \zeta) N = 0.$$  

Taking the scalar product with $N$ to this, we have $C(X, \zeta) = 0$. Replacing $X$ by $\zeta$ to this and using (4.1), we obtain $\gamma = 0$, i.e., $M$ is screen totally geodesic. Thus, by Case (1) of Corollary 1, we show that $\alpha = \beta = 0$ and $\bar{M}$ is flat.
Case (2). In case the transversal connection is flat: If $\zeta$ is tangent to $M$, then, by Case (1), we show that $\alpha = \beta = 0$ and $\bar{M}$ is flat. Thus we may assume that $\zeta$ is not tangent to $M$, i.e., $f \neq 0$.

As the transversal connection is flat, we get
\[ \beta \{ \theta(X) \eta(Y) - \theta(Y) \eta(X) \} = 0. \]
From this, we have $\beta \theta(PX) = 0$. If $\beta = 0$, then we have $\alpha = 0$ and $\bar{M}$ is flat. Thus we let $\beta \neq 0$. As $\theta(PX) = 0$, we get
\[ \zeta = e \xi + f N. \]  
(4.2)
Since $\bar{g}(\zeta, \zeta) = 1$, we get $2ef = 1$. Replacing $X$ by $\xi$ to (3.2), we get
\[ \bar{g}(\bar{R}(\xi, Y)X, N) = (\alpha + ef \beta)g(X, Y). \]  
(4.3)
Assume that $\zeta$ is parallel respect to $\nabla$. Applying $\nabla_X$ to (4.2) and using (1.4)~(1.8) and $A_NX = \gamma PX$, we show that $e$ and $f$ are constants and
\[ A^*_N X = \sigma PX, \]  
(4.4)
where $\sigma = -2f^2 \gamma$. Substituting (4.1) into (2.6) and using (2.3), (4.4) and the fact that $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$, we get
\[ \bar{g}(\bar{R}(X, Y)Z, N) = \{X[\gamma] - \sigma \gamma \eta(X)\}g(Y, Z) - \{Y[\gamma] - \sigma \gamma \eta(Y)\}g(X, Z). \]
Replacing $X$ by $\xi$ and $Z$ by $X$ to this, we have
\[ \bar{g}(\bar{R}(\xi, Y)X, N) = \{\xi[\gamma] - \sigma \gamma\}g(X, Y). \]
Comparing this with (4.3) and using $\sigma = -2f^2 \gamma$ and $2ef = 1$, we obtain
\[ 2\{\xi[\gamma] + 2f^2 \gamma^2\} = 2\alpha + \beta. \]  
(4.5)
Replacing $W$ by $\xi$ to (1.1) and using (2.2) and the fact $\tau = 0$, we have
\[ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = f \beta \{ \theta(X)g(Y, Z) - \theta(Y)g(X, Z) \}. \]  
(4.6)
Applying $\nabla_X$ to $B(Y, Z) = -2f^2 \gamma g(Y, Z)$ and using $f$ is a constant, we have
\[ (\nabla_X B)(Y, Z) = -2f^2 X[\gamma]g(Y, Z) - 2f^2 \gamma(\nabla_X g)(Y, Z). \]
Substituting this into (4.6) and using (1.9) and the fact that $f \neq 0$, we have
\[ 2f \{X[\gamma] + 2f^2 \gamma^2 \eta(X)\}g(Y, Z) - 2f \{Y[\gamma] + 2f^2 \gamma^2 \eta(Y)\}g(X, Z) \]
\[ = \beta \{ \theta(Y)g(X, Z) - \theta(X)g(Y, Z) \}. \]
Replacing $Y$ by $\xi$ to this and using the fact that $\theta(\xi) = f$, we have
\[ 2\{\xi[\gamma] + 2f^2 \gamma^2\} = -\beta. \]  
(4.7)
From (4.5) and (4.7), we get $\alpha = -\beta$. Substituting this into (3.3), we have
\[ \alpha(1 + e^2 + f^2) = 0. \]
This implies $\alpha = 0$. Thus $\beta = 0$ and $\bar{M}$ is flat.
References


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