Loubéré Magic Squares over Multi Set of Rational Numbers as an Infinite Field

A. M. Babayo
Department of Mathematics and Computer Science
Faculty of Science, Federal University Kashere, Gombe State, Nigeria

G. U. Garba
Department of Mathematics
Faculty of Science, Ahmadu Bello University, Zaria, Kaduna State, Nigeria

Abstract

This work is a pioneering investigation of the Loubéré Magic Squares over multi set of rational numbers of the form \( \left\{ \frac{1}{a} : a \neq 0, a \in \mathbb{Z} \right\} \) infinite algebraic field, where \( \mathbb{Z} \) denotes the multi set of integer numbers. By the Loubéré Magic Squares \( L \) (as we denote it), we understand the set of magic squares formed by the De La Loubéré Procedure. It is explicated that the set equipped with the matrix binary operation of addition \( \oplus \) (as we denote it) forms an infinite additive abelian group, and the set enclosed with the rational numbers multiplication \( \otimes \) (as we denote it) forms an infinite multiplicative abelian group if the underlining set so considered of the entries of the aforementioned squares is the multi set of the aforementioned set of numbers. \( (L, \oplus, \otimes) \) forms an infinite field.

Mathematics Subject Classification: 12—xx

Keywords: Loubéré Magic Squares, Multi Set, Infinite Field, Additive Abelian Group, Multiplicative Abelian Group
1. Introduction

The Loubéré Magic Squares $L$ equipped with the matrix binary operation of addition $\oplus$ forms a semigroup if the underlining set considered is the multi set of natural numbers. If we underline the multi set of integer numbers as entries of the square, $(L, \oplus)$ forms an infinite additive abelian group.

The Loubéré Magic Squares over the multi set of rational numbers of the form \(\left\{\frac{1}{a} : a \neq 0, a \in \mathbb{Z}\right\}\) forms an infinite multiplicative abelian group, $(L, \otimes)$, thus, making $(L, \oplus, \otimes)$ an infinite field. This is not the explication of the definition of the field presented in [1], but the two are analogous.

Arbitrarily, $5 \times 5$ Semi Pandiagonal Loubéré Magic Squares is considered — not squares such that $n \geq 7$ — to economy space, and not $3 \times 3$ such a square to dodge near bias choice.

The multi set of rational numbers of the form \(\left\{\frac{1}{a} : a \neq 0, a \in \mathbb{Z}\right\}\) is aptly considered even though the multi set of rational numbers of the form \(\left\{\frac{a}{b} : b \neq 0, b \text{ is fixed } \in \mathbb{Z} \text{ and } a \in \mathbb{Z}\right\}\), a special type of scalar multiplication, will do. Also, we consider the sequence $a, a, a, ... n$ times, $b, b, b, ... n$ times, $c, c, c, ... n$ times, ... rather than $a, b, c, ... n$ times, $a, b, c, ... n$ times, ... though the later will also give an analogous result; but presenting both the two is a babyish tautology.

2. Preliminaries

Definition 2.1.

A multi set is a set in which repetition of elements is relevant. For example, \(\left\{a, a, a, b, b, b, c, c, c\right\} \neq \left\{a, b, c\right\} \neq \left\{a, b, c, a, b, c, a, b, c\right\}\) for their Loubéré Magic Squares are not isomorphic.

Definition 2.2.

A basic magic square of order $n$ can be defined as an arrangement of arithmetic sequence of common difference of 1 from 1 to $n^2$ in an $n \times n$ square grid of cells such that every row, column and diagonal add up to the same number, called the magic sum $M(S)$ expressed as $M(S) = \frac{n^3 + n}{2}$ and a centre piece $C$ as $C = \frac{M(S)}{n}$.

Definition 2.3.

Main Row or Column is the column or row of the Loubéré Magic Squares containing the first term and the last term of the arithmetic sequence in the square.
2.4 Loubére Procedure (NE-W-S or NW-E-S, the cardinal points)

Consider an empty $n \times n$ square of grids of cells. Start, from the central column or row at a position $\left\lceil \frac{n}{2} \right\rceil$ where $\left\lceil \right\rceil$ is the greater integer number less than or equal to, with the number 1. The fundamental movement for filling the squares is diagonally up, right (clock wise or NE or SE) or up left (anti clock wise or NW or SW) and one step at a time. If a filled cell (grid) is encountered, then the next consecutive number moves vertically downward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S (depending on the position of the first term of the sequence) to the last row or first row or first column or last column. See also [2] for such a procedure.

The square of grid of cells $[a_{ij}]_{n \times n}$ is said to be Loubére Magic Square if the following conditions are satisfied.

i. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = k$

ii. $\text{trace}[a_{ij}]_{n \times n} = \text{trace}[a_{ij}]_{n \times n}^T = k$

iii. $a_{\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil}, a_{\left\lceil \frac{n}{2} \right\rceil, n}, a_{n, \left\lceil \frac{n}{2} \right\rceil}, a_{n, n}$ are on the same main column or row and $a_{\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil}, a_{\frac{n}{2}, \frac{n}{2}}, a_{\frac{n}{2}, 1}$ are on the same main column or row,

where $\left\lceil \right\rceil$ is the greater integer less or equal to, $T$ is the transpose (of the square), $k$ is the magic sum (magic product is defined analogously) usually expressed as $k = \frac{n}{2} [2a + (n - 1)j] - \text{from the sum of arithmetic sequence, where } j$ is the common difference along the main column or row and $a_{\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil}, a_{\frac{n}{2}, \frac{n}{2}}, a_{\frac{n}{2}, 1}$

2.5 Group

A non empty set $G$ together with an operation $*$ is known as a group if the following properties are satisfied.

i. $G$ is closed with respect to $*$. i.e., $a \ast b \in G, \forall a, b \in G$.

ii. $*$ is associative in $G$. i.e., $a \ast (b \ast c) = (a \ast b) \ast c, \forall a, b, c \in G$.

iii. $\exists e \in G$, such that $e \ast a = a \ast e = a, \forall a \in G$. Here $e$ is called the identity element in $G$ with respect to $*$.

iv. $\forall a \in G, \exists b \in G$ such that $a \ast b = b \ast a = e$, where $e$ is the identity element. Here $b$ is called the inverse of $a$ and similarly vise versa. The inverse of the element $a$ is denoted as $a^{-1}$. 
The above definition of a group is given in [1]. If in addition to the above axioms, the following axiom is satisfied, we call \((G,\ast)\) an abelian group where \((G,\ast)\) is a denotation of a group.

\[ a \ast b = b \ast a, \forall a, b \in G. \] That is all (not some of) the elements of G commutes.

2.6 Field

If \((G,\oplus)\) is an additive abelian group and \((G,\otimes)\) is a multiplicative abelian group, then \((G,\oplus,\otimes)\) is a field.

3. The Loubéré Magic Squares Field

The underlie multi set of the infinite additive Loubéré Magic Squares abelian group \((L,\oplus)\) is \(\{a: a \in \mathbb{Z}\}\) and the underlie multi set of the infinite multiplicative Loubéré Magic Squares abelian group \((L,\otimes)\) is \(\{\frac{a}{b}: a \neq 0, a \in \mathbb{Z}\}\). The whole concept is based on the manifestation of \(n \times n\) Loubéré Magic Squares where \(n\) is odd having both the magic sum and the magic product.

\{a, a, a, \ldots n\text{ times}, b, b, b, \ldots n\text{ times}, \ldots \text{or} \frac{1}{a}, \frac{1}{a}, \ldots n\text{ times}, \frac{1}{b}, \frac{1}{b}, \ldots n\text{ times}\ldots: a, b, \ldots \in \mathbb{Z}\}\) is the multi set considered, where \(n = 2\mathbb{Z}_+\)

\(1\) and \(\mathbb{Z}_+\) is the set of positive integers (greater than or equal to 1) for \(n = 1\) is a triviality and \(n = 2, \) the oddest prime, does not exist.

The corresponding set of magic squares of entries the sequence of elements of the multi set above are \(L_n\) where \(n = 3, 5, 7, \ldots\)

\[
L_3 := \left\{% \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\},
\]
Loubére magic squares over multi set of rational numbers

\[ L_5 = \left\{ \begin{array}{c} d \ a \ b \ c \\ e \ a \ b \ c \ d \\ a \ b \ c \ d \ e \\ b \ c \ d \ e \ a \\ c \ d \ e \ a \ b \end{array} \right\} \text{ or } \begin{pmatrix} 1 & 1 & 1 & 1 \\ d & e & a & b \\ 1 & 1 & 1 & 1 \\ e & a & b & c \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \\ b & c & d & e \\ 1 & 1 & 1 & 1 \\ c & d & e & a \\ b & c & d & e \\ a & b & c & d \\ e & a & b & c \end{pmatrix} : a, b, c \in \mathbb{Z} \]

\[ L_7, L_9, L_{11}, \ldots \] are constructed analogously.

Theorem 3.1. \((L_n, \oplus)\) is an additive abelian group.

Proof. Arbitrarily considering \(L_5\), we define \(\oplus\) as follows: Let \(A, B \in L_5\) where

\[ A = \begin{pmatrix} d & e & a & b & c \\ e & a & b & c & d \\ a & b & c & d & e \\ b & c & d & e & a \\ c & d & e & a & b \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x & y & u & v & w \\ y & u & v & w & x \\ u & v & w & x & y \\ v & w & x & y & u \\ w & x & y & u & v \end{pmatrix} \]

\[ A \oplus B = \begin{pmatrix} d & e & a & b & c \\ e & a & b & c & d \\ a & b & c & d & e \\ b & c & d & e & a \\ c & d & e & a & b \end{pmatrix} \oplus \begin{pmatrix} x & y & u & v & w \\ y & u & v & w & x \\ u & v & w & x & y \\ v & w & x & y & u \\ w & x & y & u & v \end{pmatrix} = \begin{pmatrix} d + x & e + y & a + u & b + v & c + d + x \\ e + y & a + u & b + v & c + d + x & e + x \\ a + u & b + v & c + d + x & e + y & a + u \\ b + v & c + d + x & e + y & a + u & b + v \\ c + d + x & e + y & a + u & b + v & c + d + x \end{pmatrix} \]

i. \(L_5\) is closed with respect to \(\oplus\): From the above definition, if \(A, B \in L_5\), then \(A \oplus B = C \in L_5\). This is more vivid by letting (say) \(p = a + u, q = b + v, r = c + w, s = d + x\) and \(t = e + y\).

ii. \(\oplus\) is associative: For \(A, B, C \in L_5\), \(A \oplus (B \oplus C) = (A \oplus B) \oplus C\) whence \(i + (j + (i + j)) = (i + j) + (i + j)\) where \(i \in A, j \in B\) and \(i + j \in C\) (arbitrarily chosen.)
iii. The additive identity element \( I \) is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
where each 0 except the first entry of the square is a series of 0s correspondingly \( n \) times.

iv. Each element of \( L_5 \) has an inverse: If \( D \in L_5 \), then \( -D \in L_5 \) where \( -D \) is the square with magic sum \( -M(S) \) formed as a result of scalar multiplication of entries in \( D \) having magic sum \( M(S) \) by \( -1 \).

For example, the inverse of
\[
\begin{bmatrix}
b & a & c \\
c & b & a \\
a & c & b
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
-b-a & -c & 0 \\
-c-b & -a & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

v. \( \oplus \) is commutative: \( A, B \in L_5 \Rightarrow A \oplus B = B \oplus A \) whence the matrix binary operation of addition over set of integer numbers is commutative (inheritance).

Thus, \( (L_n, \oplus) \) is an infinite additive abelian group.

**Theorem 3.2.** \( (L_n, \otimes) \) is an infinite multiplicative abelian group.

**Proof.** Let \( D, E \in L_5 \). We define \( \otimes \) as follows: Let \( D =\begin{bmatrix}
l & m & i & j & k \\
m & i & j & k & l \\
i & j & k & l & m \\
j & k & l & m & i \\
k & l & m & i & j
\end{bmatrix} \) and
\[
\begin{bmatrix}
h & i & e & f & g \\
i & e & f & g & h \\
e & f & g & h & i \\
f & g & h & i & e \\
g & h & i & e & f
\end{bmatrix} = E.
\]
Then \( D \otimes E =\begin{bmatrix}
lh & mi & ie & jf & kg \\
mi & ie & jf & kg & lh \\
ie & jf & kg & lh & mi \\
fj & kg & lh & mi & ie \\
kg & lh & mi & ie & jf
\end{bmatrix} = F \)

i. \( L_5 \) is closed with respect to \( \otimes \): For \( D, E \in L_5, F \in L_5 \) from the above definition. This is vivid by intimate look at the pattern of elements in \( F \): ie, jf, kg, lh, and mi.

ii. Associativity: \( \otimes \) is associative for \( A, B, C \in L_5 \Rightarrow A \otimes (B \otimes C) = (A \otimes B) \otimes C \) because \( x(y(xy)) = (xy)(xy) \) where \( x \in A, y \in B \) and \( xy \in C \). This is \( \forall x, y \) and \( xy \).
iii. The identity element I is \[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
where each 1 \( \in I \) except the first entry of the square is a product of 1s correspondingly n times.

iv. Each element of \( L_5 \) has an inverse: If \( G = \begin{bmatrix}
i & j & f & g & h \\
j & f & g & h & i \\
f & g & h & i & j \\
g & h & i & j & f \\
h & i & j & f & g
\end{bmatrix} \in L_5 \), then \( \exists H = \begin{bmatrix}
i & j & f & g & h \\
j & f & g & h & i \\
f & g & h & i & j \\
g & h & i & j & f \\
h & i & j & f & g
\end{bmatrix} \) such that \( G \otimes H = H \otimes G = I \)

v. \( \otimes \) is commutative: \( A, B \in L_5 \Rightarrow A \otimes B = B \otimes A \) since the entries in the square are rational numbers and rational numbers multiplication is commutative, we are done.

Thus, \( (L_n, \otimes) \) is an infinite multiplicative abelian group.

**Theorem 3.3.** \( (L_n, \oplus, \otimes) \) is an infinite field.

**Proof.** This follows immediately from Theorem 3.1 and Theorem 3.2 above.

---

**4. Conclusion**

Every Loubéré Magic Square under discuss in this work has about 4 miscellany effects of rotations and/or reflections. Considering 1 out of the 4 effects is arbitrary. Although the Loubéré Magic Squares presented here are not the basic (obvious) ones, yet they are semi pandiagonal Loubéré — with unique magic sums and products. We recommend that if you are apt in searching for an example of any algebraic structures be it semigroup (Fibonacci), group (Symmetric), field (Loubéré), vector space(Magic Squares in general) or not enter Loubéré Magic Squares, you will find one.
References


Received: November 8, 2014; Published: December 12, 2014