On RG – Algebra

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Abstract: In this paper we introduce the notion of RG-algebra which is a good generalization of some algebraic structures such as the medial (BCI\BCH-algebras), the BM and BF-algebras.

Keywords: congruences, ideal, homomorphism, BCK\BCI – algebras

1. Introduction

Iami and Iseki introduced the notions of the two algebraic structures BCK – algebra and BCI – algebra [2, 12]. BCK-algebra is now known as a proper subclass of the class of BCI – algebra. Hu and Li [8, 9] introduced the notion of BCH – algebra. Neggers and Kim [13] introduced the notion of d-algebra which is another generalization of BCK – algebra – Jun, Roh, and Kim [15] introduced the notion of BH – algebra which is a generalization of (BCK \ BCI) – algebras. In the present paper we introduce what we call RG – algebra which is a good generalization of the previous algebraic structures and study some of its basic properties. The purpose of this paper is derive some straightforward consequences relations between the RG-algebra and the abelian group which is related to it.

2. Preliminaries

In the sequel we introduce some needed definitions and results.
Definition (2.1): A BCI – algebra [5] is an algebra \((X; *, 0)\) of type \((2, 0)\) satisfying the following conditions: -
(i) \((x \ast y) \ast (x \ast z) \ast (z \ast y) = 0\)
(ii) \((x \ast (x \ast y)) \ast y = 0\)
(iii) \(x \leq x\).
(iv) \(x \leq y\) and \(y \leq x\) imply \(x = y\).
(v) \(x \leq 0\) implies \(x = 0\).

where \(x \leq y\) is defined by \(x \ast y = 0\). If (v) is replaced by \(0 \leq x\), \(\forall x \in X\) then the algebra is called BCK-algebra.

**Definition (2.2):** A BCH-algebra [9] is an algebra of type \((2, 0)\) satisfying the following conditions:

i) \(x \ast x = 0\).
ii) \(x \leq y\) and \(y \leq x\) imply \(x = y\).
iii) \((x \ast y) \ast z = (x \ast z) \ast y\), where \(x \leq y\) is and only if \(x \ast y = 0\).

**Definition (2.3):** A BCH-algebra is proper (cf [9]) if and only if does not satisfy the condition.

\[(x \ast y) \ast (x \ast z) \leq z \ast y\]

### 3. RG – algebra

In this article we introduce the notion of RG – algebra with some important results related to it.

**Definition (3.1):** An algebra \((X; \ast, 0)\) is called RG – algebra if the following axioms are satisfied:

i) \(x \ast 0 = x\).
ii) \(x \ast y = (x \ast z) \ast (y \ast z)\) \(\forall x, y, z \in X\)
iii) \(x \ast y = y \ast x = 0\) imply \(x = y\).

By (ii) above put \(y = 0\) we get \(x \ast 0 = (x \ast z) \ast (0 \ast z)\), also put \(x = 0\), then \(0 \ast 0 = (0 \ast z) \ast (0 \ast z)\) by (i) we have \(0 = (0 \ast z) \ast (0 \ast z)\), let \(0 \ast z = x\) then \(0 = x \ast x\), \(\forall x \in X\) also \((x \ast y) \ast z = (x \ast y) \ast (z \ast 0) = (x \ast z) \ast (y \ast 0) = (x \ast z) \ast y\).

**Example (3.2):** Let \(X = \{0, a, b, c\}\) and \((X, \ast)\) be the pair given by the table.

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</table>

Then \((X; \ast, 0)\) is an RG – algebra.

**Definition (3.3):** A \((BCH \setminus BCI)\) – algebra is called medial (cf. [1]) if

\[(x \ast y) \ast (z \ast \mu) = (x \ast z) \ast (y \ast \mu), \forall x, y, z, \mu \in X.\]

**Theorem (3.4):** A \((BCH \setminus BCI)\) – algebra is medial if and only if

\[x \ast y = (x \ast z) \ast (y \ast z), \forall x, y, z \in X.\]

**Proof:** Let \(X\) be a medial \((BCH \setminus BCI)\) – algebra, then
\[(x \ast y) \ast (z \ast \mu) = (x \ast z) \ast (y \ast \mu), \quad \text{put } z = \mu\]

\[(x \ast y) \ast (z \ast z) = (x \ast z) \ast (y \ast z)\]

\[(x \ast y) \ast 0 = (x \ast z) \ast (y \ast z), \quad \text{then}\]

\[(x \ast y) = (x \ast z) \ast (y \ast z)\]

Conversely, let \((x \ast y) = (x \ast \mu) \ast (y \ast \mu), \quad \forall x, y, \mu \in X\), then

\[(x \ast y) \ast (z \ast \mu) = (((x \ast y) \ast 0) \ast (z \ast \mu)) = ((x \ast \mu) \ast (y \ast \mu)) \ast (z \ast \mu) = (x \ast z) \ast (y \ast \mu)\]

That is \(X\) is medial.

**Proposition (3.5):** In any RG-algebra the following hold:

i) \(0 \ast (y \ast x) = x \ast y\).

ii) \(0 \ast (0 \ast x) = x\).

iii) \(x \ast (x \ast y) = y\)

iv) \(x \ast y = (z \ast y) \ast (z \ast x) \quad \forall x, y, z \in X\).

v) \(x \ast y = 0 \quad \text{if and only if } y \ast x = 0\).

**Proof:**

i) \(0 \ast (y \ast x) = (0 \ast y) \ast (0 \ast x) = ((x \ast x) \ast y) \ast (0 \ast x) = (x \ast y) \ast (x \ast x) \ast 0 = x \ast y\).

ii) Put \(y = 0\) in (i) above \(0 \ast (0 \ast x) = x \ast 0 = x\).

iii) \(x \ast (x \ast y) = (x \ast 0) \ast (x \ast y) = (x \ast x) \ast (0 \ast y) = 0 \ast (0 \ast y) = y\).

iv) Using (i) above we have \(x \ast y = 0 \ast (y \ast x) = (z \ast z) \ast (y \ast x) = (z \ast y) \ast (z \ast x)\).

v) If \(x \ast y = 0\) then \(y \ast x = 0 \ast (x \ast y) = 0 \ast 0 = 0\), the converse is similar.

**Proposition (3.6):** Every RG-algebra is a BCI-algebra.

**Proof:** \(\forall x, y, z \in X\) we have:

\[((x \ast y) \ast (x \ast z)) \ast (z \ast y) = ((x \ast y) \ast (y \ast z)) \ast (z \ast y)\]

\[= 0 \ast (y \ast z) \ast (z \ast y)\]

\[= (z \ast y) \ast (z \ast y) = 0\]

Then \(X\) is a BCI-algebra.

The converse of this proposition may be not true.

**Example (3.7):** Let \(X = \{0, a, b, c\}\) in which \(*\) is defined by

<table>
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<td>c</td>
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</table>
Then \((X; *, 0)\) is a BCI–algebra but it is not an RG–algebra because \(0 * (b * c) = 0, \ c * b = a\) then \(0 * (b * c) \neq c * b\).

**Example (3.8):** Let \(X = \{0, 1, 2, 3\}\) in which \(*\) is defined by

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<th>1</th>
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<th>3</th>
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<td>0</td>
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<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</table>

Then \((X; *, 0)\) is a BCH–algebra but is not an RG–algebra since \((1 * 3) * (2 * 3) = 3 * 2 = 0 \neq 3 = (1 * 2)\).

**Proposition (3.9):** In any RG – algebra \(X\) the following hold:

i) \((x * y) * (0 * y) = (x * (0 * y)) * y = x.\)

ii) \(x * (x * (x * y)) = x * y.\)

iii) \((x * y) * z = (x * y) * ((z * y) * (0 * y)) = ((x * z) * z) * (y * z) = ((x * y) * y) * (z * y) = (x * z) * y.\)

**Proof:** i) Since \(x = x * 0 = (x * 0) * (y * y) = (x * y) * (0 * y) = (x * (0 * y)) * y.\)

ii) From proposition (3.4), (iii) since \(y = x * (x * y),\) then \(x * y = x * (x * (x * y)).\)

iii) From (i) above; \(z = (z * y) * (0 * y),\) then \((x * y) * z = (x * y) * ((z * y) * (0 * y)),\) also \(((x * z) * z) * (y * z) = ((x * z) * (y * z)) * z = (x * y) * z.\) Finally \(((x * y) * y) * (z * y) = ((x * y) * (z * y)) * y = (x * z) * y = (x * y) * z.\)

**Theorem (3.10):** Let \((G, \lambda)\) be an abelian group then \((G, *, e)\) is an RG – algebra where \(e\) is the identity for the operation \(\lambda\) do as the zero of the operation \(*\) and \(x * y = x \lambda y^{-1}, \ \forall x, y \in G.\)

**Proof:** Let \(x, y, z \in G.\)

i) \(x * 0 = x \lambda e^{-1} = x \lambda e = x.\)

ii) Let \(x * y = y * x = 0.\) That is \(x \lambda y^{-1} = y \lambda x^{-1} = e,\) then \(x = e \lambda x = (y \lambda x^{-1}) \lambda x = y \lambda (x^{-1} \lambda x) = y \lambda e = y\)

iii) \((x * y) * (y * z) = (x \lambda z^{-1}) \lambda (y \lambda z^{-1})^{-1} = (x \lambda z^{-1}) \lambda (z \lambda y^{-1}) = x \lambda (z^{-1} \lambda z) \lambda y^{-1} = x \lambda e \lambda y^{-1} = x \lambda y^{-1} = x * y.\)

**Corollary (3.11):** Every \((Z_n; *, [0])\) is an RG – algebra.
\textbf{Proof:} From the last theorem take the operation $\ast$ as the inverse of the addition mod $n$ and $[0]$ stands as the zero in this algebra.

\textbf{Theorem (3.12):} Let $(X; 0, 0)$ be an RG – algebra in which $x \ast y \neq 0 \ \forall x \neq y$ in $X$ then the system $(X, \lambda)$ is an abelian group where the operation $\lambda$ is defined as:

\[ x \lambda y = x \ast (0 \ast y) \ \forall x, y \in X. \]

\textbf{Proof:} (i) \[ x \lambda (y \lambda z) = x \lambda (y \ast (0 \ast z)) = x \ast (0 \ast (y \ast (0 \ast z))) = x \ast ((0 \ast y) \ast (0 \ast z))) \]

and

\[ (x \lambda y) \lambda z = (x \ast (0 \ast y)) \ast (0 \ast z) = (x \ast 0) \ast ((0 \ast y) \ast y) = x \ast ((0 \ast y) \ast z). \]

That is \[ (x \lambda y) \lambda z = x \lambda (y \lambda z) \]

which gives that $\lambda$ is an associative.

ii) since $e \lambda x = 0 \ast (0 \ast x) = x$ and $x \lambda e = x \ast (0 \ast 0) = x \ast 0 = x$. So $0 \lambda x = x \lambda 0 = x$ which gives that the zero of the operation $\ast$ play the role as the identity $e$ for the operation $\lambda$.

iii) The inverse of $x \in X$ for the operation $\lambda$ in $(X; 0, 0)$ is the element $(0 \ast x)$ since

\[ x \lambda (0 \ast x) = x \ast (0 \ast (0 \ast x)) = x \ast x = 0 = e \]

and $(0 \ast x) \lambda x = (0 \ast x) \ast (0 \ast x) = 0 = e$.

iv) \[ (x \lambda y) \ast (y \lambda x) = (x \ast (0 \ast y)) \ast (y \ast (0 \ast x)) \]

\[ = (x \ast y) \ast ((0 \ast y) \ast (0 \ast x)) \]

\[ = (x \ast y) \ast (x \ast y) = 0 \quad \text{and} \]

\[ (y \lambda x) \ast (x \lambda y) = (y \ast (0 \ast x)) \ast (x \ast (0 \ast y)) \]

\[ = (y \ast x) \ast (0 \ast (x \ast y)) = (y \ast x) \ast (y \ast x) = 0 \]

Then $x \lambda y = y \lambda x$. Therefore $(x, \lambda)$ is an abelian group.

\textbf{Example (3.13):} Let $X = \{0, 1, 2, 3, 4, 5\}$ and the pair $(X; 0, 0)$ be given by the following table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
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</table>

It is an easy to verify that $(X; 0, 0)$ is an RG – algebra.

\textbf{Example (3.14):} The group $(X, \lambda)$ according to theorem (3.12) arising from the RG – algebra in example (3.2) is given by the table:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>e</th>
<th>a</th>
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<tr>
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<td>e</td>
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</table>
Definition (3.15): A BCI – algebra \((X; *, 0)\) (cf. [1]) is called:
i) Weakly positive implicative if \((x * y) * z = ((x * z) * z) * (y * z)\).
ii) Weakly implicative if \((x * (y * x)) * (0 * (y * x)) = x\) as well as BCH – algebra.
iii) Weakly commutative if \((x * (x * y)) * (0 * (x * y)) = y * (y * x)\) as well as BCH – algebra.

While a BCH– algebra is called weakly positive implicative if \(x * y = ((x * y) * y) * (0 * y)\).

Proposition (3.16): Let \(X\) be an RG – algebra then the following hold:
i) \(X\) is weakly positive implicative.
ii) \(X\) is weakly commutative.
iii) \(X\) is weakly implicative.

Proof: (i) Comes soon from (3.9) if we regard \(X\) as a BCI – algebra and if we regard \(X\)
as a BCH – algebra from (3.9) (i); \(x = (x * y) * (0 * y)\), then \(x * y = ((x * y) * (0 * y)) * y = ((x * y) * y) * (0 * y)\) that is \(X\) is weakly positive implicative.

ii) Since \((x * (x * y)) * (0 * (x * y)) = ((x * 0) * (x * y)) * (0 * (x * y))\)

\[= ((x * x) * (0 * y)) * (0 * (x * y)) = (0 * (0 * y)) * (0 * (x * (x * y)))\]

\[= y * (y * x).\]
Therefore \(X\) is weakly commutative.

iii) Since \((x * (y * x)) * (0 * (y * x)) = (x * 0) * ((y * x) * (y * x)) = x * 0 = x.\)
Therefore \(x\) is weakly commutative.

4. Ideals and congruences in RG – algebra

In this article we introduce a definition of ideals in RG – algebra we notice that it is equivalent to the well known concept of ideals in (BCK \ BCI) – algebras and study the connections between such ideals and congruences.

Definition (4.1): A non – empty subset \(A\) of a (BCK \ BCI) – algebra \((X; *, 0)\) is called an ideal of \(X\) [cf [10, 11]] if:
i) \(0 \in A.\)
ii) \(x * y \in A\) and \(y \in A\) imply \(x \in A.\)

Definition (4.2): Let \((X; *, 0)\) be an RG – algebra, a non empty subset \(A\) of \(X\) is called an ideal of \(X\) if:
i) \(0 \in A.\)
ii) \(x * y \in A\) and \(0 * x \in A\) imply \(0 * y \in A\), \(\forall x, y \in X.\)

If \(A\) is an ideal in the RG – algebra \(X\), then the relation \(0\) on \(X\) defined by \(x0y\) if and only if \(x * y, y * x \in A\) is called the relation defined by the ideal \(A.\)

Lemma (4.3): In an RG – algebra \(X\) every RG – ideal is a BCK – ideal.

Proof: Let \(A\) be an ideal in an RG – algebra \(X\), then \(0 \in A\) and if \(x * y \in A, y \in A\) then \(0 * y \in A\) and \(x = x * 0 = x * (y * y) = (x * y) * (0 * y)\) since \(x * y, 0 * y \in A\), then \(x \in A.\) That is \(A\) is a BCK – ideal.

Lemma (4.4): In a BCK – algebra any BCK – ideal is an RG – ideal.
On RG – algebra

Proof: Let \( A \) be a BCK – ideal in the BCK–algebra \( X \), then \( 0 \in A \) and whenever \( x \ast y \in A \), then \( 0 \ast x = 0 \ast y = 0 \), so \( 0 \ast x, 0 \ast y \in A \), that is \( A \) is an RG – ideal.

Lemma (4.5): In any RG – algebra any BCK – ideal is an RG – sub algebra.

Proof: Let \( A \) be a BCK – ideal in an RG – algebra \( X \), then \( 0 \in A \) and \( \forall x, y \in X; (x \ast y) \ast x = 0 \ast y \). Thus for all \( x, y \in A \) then \( (x \ast y) \ast x = 0 \ast y \in A \), now since \( x \in A \), \( 0 \ast y \in A \), then \( x \ast y \in A \). Therefore \( A \) is an RG – sub algebra of \( X \).

Corollary (4.6): Any RG – ideal in an RG – algebra \( X \) is an RG – sub algebra of \( X \).

Proof: Comes soon from (4.3) and (4.5).

Theorem (3.7): Let \( A \) be an ideal in an RG – algebra \( X \), if \( x0y \in A \) and \( x \in A \), then \( y \in A \).

Proof: Let \( x0y \) then \( x \ast y, y \ast x \in A \). For any \( x \in X; x = x \ast 0 = x \ast (y \ast y) = (x \ast y) \ast (0 \ast y) \). Now if \( x \in A \), then \( x = (x \ast y) \ast (0 \ast y) \in A \) but since \( y \ast x \in A \), \( y \ast x = 0 \ast (x \ast y) \in A \), then \( 0 \ast (0 \ast y) = y \in A \).

Theorem (3.8): Let \( A \) be an ideal in an RG – algebra \( X \), then the relation defined in (4.2) is a congruence on \( X \).

Proof: It is clear that the relation \( 0 \) defined on \( X \) is reflexive and symmetric. To show that it is transitive let \( x0y \) and \( y0z \) then \( x \ast y, y \ast z, z \ast y \in A \), but \( x \ast z = (x \ast y) \ast (z \ast y) \) and \( z \ast x = (z \ast y) \ast (x \ast y) \) then using (4.6) we get \( x \ast z, z \ast x \in A \). That is \( \theta \) is transitive and hence \( \theta \) is an equivalence relation.

Now let \( x0a, y0b \) then \( x \ast a, a \ast x, y \ast b, b \ast y \in A \) but since \( X \) is an RG – algebra then \( (x \ast y) \ast (a \ast b) = (x \ast a) \ast (y \ast b) \) and \( (a \ast b) \ast (x \ast y) = (a \ast x) \ast (b \ast y) \). This gives that \( (x \ast y) \ast (a \ast b), (a \ast b) \ast (x \ast y) \in A \) hence \( (x \ast y) \theta (a \ast b) \). Therefore \( \theta \) is a congruence.

If the relation \( 0 \) is a congruence on an RG – algebra \( X \) then \( C_0 = \{ y \in X : x0y \} \) is the equivalence class of \( x \in X \) and the family \( \{ C_x : x \in X \} \) form a partition of \( X \) which is always denote by \( X \) \( \theta \). On \( x \theta 0 \) we define \( C_x \ast C_y = C_{xy} \forall x, y \in X \). Since \( 0 \) has the substitution property then the operation \( 0 \) is well defined on \( x \theta 0 \). It is easy to verify that \( (x \theta 0, \ast, C_0) \) satisfies all the axioms of the RG – algebra except (1.3), (iii). If the \( x \theta 0 \) holds for all the classes \( C_y \in x \theta 0 \) that is if the system \( (x \theta 0, \ast, C_0) \) is an RG – algebra then the congruence \( 0 \) is called regular.

Theorem (4.9): if \( 0 \) is a congruence on an RG – algebra \( X \) then \( C_0 = \{ x \in X : x00 \} \) is an ideal of \( X \).

Proof: It is clear that \( 0 \in C_0 \), let \( x,y \in X \) be such that \( x \ast y, 0 \ast x \in C_0 \) then \( (0, 0) \in 0 \), \( (x \ast y, 0) \) \( 0 \in 0 \) but since \( y \ast x = 0 \ast (x \ast y) \) then \( (y \ast x, 0) \in 0 \). Now \( (0 \ast x) \ast (y \ast x)00 \) which give \( (0 \ast y) \ast (x \ast x)00 \) that is \( (0 \ast y)00 \). So \( 0 \ast y \in C_0 \) and hence \( C_0 \) is an ideal of \( X \).

Note that \( C_0 = A \) for any congruence \( 0 \) defined by the relation that mentioned in (4.2). Then as a consequence of this result we get:

Corollary (4.10): Any ideal in any RG – algebra \( X \) can be determined by some congruence.
Corollary (4.11): The lattice of all congruences of an RG – algebra \( X \) is a complete lattice where the least one is defined by the ideal \( \{0\} \) and the greatest by all \( X \).

Theorem (4.12): A congruence on an RG – algebra \( X \) is regular if and only if it is defined by some RG – ideal.

Proof: Let \( \theta_A \) be a congruence defined by an RG – ideal \( A \). Then \( A_0 = A \) and \( A_{x+y} = A_0 = A_{x+y} \) thus \( x \ast y, y \ast x \in A \) which means that \( x \theta_A y \) and \( A_x = A_y \). Therefore the congruence defined by the ideal \( A \) is regular.

Now let \( \theta \) be an arbitrary regular congruence and \( x \theta y \) then \( (x \ast y)00 \) and \( (y \ast x)00 \) but since \( \theta \) is reflexive then \( C_{x+y} = C_0 = C_{y+x} \) and \( x \ast y, y \ast x \in C_0 \) with \( C_0 = A \) is an ideal in the RG – algebra \( X \) that is \( \theta \leq \theta_A \).

Conversely let \( (x \ast y) = (y \ast x) \) then \( x \ast y, y \ast x \in A = C_0 \) and \( C_x \ast C_y = C_0 = C_y \ast C_x \) which implies that \( C_x = C_y \) because \( \theta \) is regular thus \( x \theta y \).

Corollary (4.13): All congruences of a finite RG – algebra are regular and the theory of universal algebra yields.

Theorem (4.14): if \( \rho, \sigma \) are two congruences on an RG – algebra \( X \) then \( \rho \theta \sigma \) is a congruence on \( X \) if and only if \( \rho \ast \sigma = \sigma \ast \rho \).

5. Special elements in an RG – algebra

Here we introduce some special elements which satisfy a certain conditions in the RG – algebra and study some of their important properties.

Definition (5.1): An element \( a \) in an RG – algebra \( X \) is called a medial element if it satisfies the condition \((x \ast a) \ast x = a, \forall x \in X\). The set of all medial elements in \( X \) is denoted by \( M(X) \).

Proposition (5.2): Let \( X \) be an RG – algebra and \( M(X) \) be the set mentioned above then the following hold:

i) \( 0 \in M(X) \).

ii) \( a \in M(X) \) if and only if \((x \ast a) \ast x \in M(X)\).

iii) \( a \in M(X) \) if \( 0 \ast a = a \).

iv) \( a \in M(X) \) if \( (a \land x) \in M(X) \).

v) If \( a \in M(X) \) then \( a \ast (0 \ast y) = y \ast a \) \( \forall y \in X \).

vi) If \( a, b \in M(X) \) then \( a \ast b = b \ast a \).

Proof: (i) since \( \forall x \in X \) we have \((x \ast 0) \ast x = (x \ast x) \ast 0 = 0 \ast 0 = 0 \) then \( 0 \in M(X) \).

(ii) Let \( a \in M(X) \), then \((x \ast a) \ast x = a \) and \( \forall y \in X \) we have \((y \ast ((x \ast a) \ast x)) \ast y = (y \ast y) \ast ((x \ast a) \ast x) = 0 \ast (0 \ast a) = a = (x \ast a) \ast x \), this give \((x \ast a) \ast x \in M(X) \)

Conversely let \((x \ast a) \ast x \in M(X) \), then \( \forall y \in X \) we have:

\((y \ast (x \ast a) \ast x) \ast y = (x \ast a) \ast x \ast a = (x \ast a) \ast x, \) then \( a \in M(X) \).

(iii) Let \( a \in M(X) \), then from the definition \((x \ast a) \ast x = a, \forall x \in X \) that is \( 0 \ast a = a \).
Conversely let \( 0 \ast a = a \), then \( \forall x \in X \) we have 
\[(x \ast a) \ast x = (x \ast (0 \ast a)) \ast x = (x \ast x) \ast (0 \ast a) = 0 \ast (0 \ast a) = a\]
that is \( a \in M(X) \).

iv) Let \( a \in M(X) \), then \( \forall y \in X \) we have 
\[(y \ast (a \land x)) \ast y = (y \ast (x \ast (x \ast a))) \ast y = (y \ast y) \ast (x \ast (x \ast a))\]
\[= 0 \ast (x \ast (x \ast a)) = (x \ast a) \ast x = a\]
But \( a \in M(X) \), then \((y \ast (a \land x)) \ast y) \in M(X) \). According to (ii) above we have \( a \land x \in M(X) \).

Conversely let \( a \land x \in M(X) \), then 
\[(y \ast (a \land x)) \ast y = a \land x = x \ast (x \ast a) = (x \ast x) \ast (0 \ast a) = 0 \ast (0 \ast a) = a\]
But \( (a \land x) \in M(X) \), then \( a \in M(X) \).

v) Let \( a \in M(X) \), then \( a \ast (0 \ast y) = (0 \ast a) \ast (0 \ast y) = (0 \ast 0) \ast (a \ast y) = 0 \ast (a \ast y) = y \ast a \).

vi) Let \( a, b \in M(X) \), then \( 0 \ast a = a, 0 \ast b = b \), then 
\[a \ast b = (0 \ast a) \ast (0 \ast b) = (0 \ast 0) \ast (a \ast b) = 0 \ast (a \ast b) = b \ast a.\]

**Definition (5.4):** A BCI–algebra \( X \) is called quasi right alternate (cf [12]) if \( x \ast (y \ast y) = (x \ast y) \ast y \).

**Lemma (5.5):** Let \( X \) be an RG – algebra such that \( M(X) = X \) then \( X \) is a quasi right alternate. Moreover \( x \ast y = y \ast x \ \forall x, y \in X \).

**Proof:** Let \( X \) be an RG - algebra and \( M(X) = X \) for any \( x, y \in X \); \( x, y \in M(X) \) then 
\[x \ast (y \ast y) = (x \ast 0) \ast (y \ast y) = (x \ast y) \ast (0 \ast y) = (x \ast y) \ast y.\]

Similarly \( y \ast (x \ast y) = (y \ast x) \ast x \). That is \( X \) is a quasi right alternate and \( x \ast y = (0 \ast x) \ast (0 \ast y) = 0 \ast (x \ast y) = y \ast x \).

It is well noting that the RG – algebra \((X; \ast, 0)\) constructed in example (3.2) is an algebra satisfying \( M(X) = X \).

**Theorem (5.6):** Let \((X, \ast, 0)\) be an RG – algebra then the set \( M(X) \) is an RG – ideal in \( X \).

**Proof:** Clearly \( 0 \in M(X) \) let if \( x, y \in X \) be two elements such that \( x \ast y, 0 \ast x \in M(X) \), then \( 0 \ast x = x \in M(X) \) and \( 0 \ast y = (x \ast x) \ast y = (x \ast y) \ast x = (0 \ast (x \ast y)) \ast (0 \ast x) = (y \ast x) \ast (0 \ast x) = y \ast (x \ast x) = y \ast 0 = y \), that is \( y \in M(X) \), but since \( 0 \ast y = y \) then \( 0 \ast y \in M(X) \) and hence \( M(X) \) is an ideal of \( X \).

**Proposition (5.7):** Let \((X; \ast, 0)\) be an RG – algebra then the set \( M(X) \) is a sub – algebra of \( X \).

**Proof:** Let \( M(X) \) be the set of all medial elements in the RG-algebra \( X \) and \( x, y \in X \) be such that \( x, y \in M(X) \), then \( 0 \ast x = x \) and \( 0 \ast y = y \), now 
\[0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) = x \ast y.\]
That is \( x \ast y \in M(X) \) and hence \( M(X) \) is a subalgebra of \( X \).

**Lemma (5.8):** Let \((X; \ast, 0)\) be an RG – algebra such that \( M(X) = X \) then \( \forall x \in X \) the subset \( A = \{0, x\} \) is an RG – ideal (subalgebra) of \( X \).

**Theorem (5.9):** Let \((X; \ast, 0)\) be an RG – algebra, then \( 0 \ast x = x, \forall x \in X \) if and only if \( x \ast (y \ast z) = (x \ast y) \ast z \ \forall x, y, z \in X \) and \((X; \ast, 0)\) is a group in which every element is involution.

**Proof:** Let \( x \ast (y \ast z) = (x \ast y) \ast z \ \forall x, y, z \in X \), take \( x = y = 0 \), the \( 0 \ast (0 \ast 0) = (0 \ast 0) \ast 0 \) then \( z = 0 \ast z \), conversely if \( 0 \ast x = x \), then
(x * y) * z = (x * y) * (0 * z) = (x * 0) * (y * z) = x * (y * z).

**Definition (5.10):** Two elements \( x, y \) in an RG-algebra \((X; \ast, 0)\) are said to be conjugate to each other if they satisfy the condition \((x \ast y) \ast x = x\). The collection of all elements that conjugate to an element \( x \in X \) is denoted by \( C(x) \). From this definition it is clear that if \( y \in C(x) \) then \( 0 \ast y = x \).

**Lemma (5.11):** Let \((X; \ast, 0)\) be an RG – algebra the following statements hold:

i) \( 0 \in C(0) \).

ii) \( y \in C(x) \) if and only if \( x \in C(y) \).

iii) \( x \in M(X) \) if and only if \( x \in C(x) \).

iv) \( x \in C(y) \) implies to \( x \wedge y \in C(y) \).

v) \( x \in C(y) \) and \( y \in C(z) \) then \( x = z \).

**Proof:** (i) since \( \forall x \in X; (x \ast 0) \ast x = 0 \), then \( 0 \in C(0) \).

(ii) Let \( y \in C(x) \), then \( 0 \ast y = x \) and \( 0 \ast x = 0 \ast (0 \ast y) = y \) that is \( x \in C(y) \). Similarly \( x \in C(y) \) then \( y \in C(x) \).

(iii) Let \( x \in M(X) \), then \( 0 \ast x = x \) and hence \( x \in C(x) \). Conversely let \( x \in C(x) \), then \((x \ast x) \ast x = x\), so \( 0 \ast x = x \) which give \( x \in M(X) \).

(iv) Let \( x \in C(y) \) then \( 0 \ast x = y \), so \( 0 \ast (x \wedge y) = 0 \ast (y \ast (y \ast x)) = (y \ast x) \ast y = (0 \ast x) = y \). Therefore \( x \wedge y \in C(y) \).

(v) Let \( x \in C(y) \) and \( y \in C(z) \) then \( 0 \ast x = y \) and \( 0 \ast y = z \), now \( x = 0 \ast (0 \ast x) = 0 \ast y = z \).

6. Homomorphism and some maps on RG – algebras

**Definition (1.6):** Let \((X; \ast, 0), (Y, \ast, 0)\) be two RG – algebras. A mapping \( f : X \rightarrow Y \) is called a homomorphism if \( f(x \ast y) = f(x) \ast f(y) \); \( x, y \in X \) and is called an antihomomorphism if \( f(x \ast y) = f(y) \ast f(x) \). If \( f \) is a homomorphism then \( \ker f = \{ x \in X : f(x) = 0 \} \) and \( f(0) = 0 \).

**Lemma (6.2):** Let \( f : X \rightarrow Y \) be an epimorphism of RG – algebras then \( f \) maps medial elements (conjugate elements) in \( X \) onto medial (conjugate) elements in \( Y \).

**Proof:** Let \( a \in M(X) \), then \((x \ast a) \ast x = a, \forall x \in X \) and \( f((x \ast a) \ast x) = f(x \ast a) \ast f(x) = f(a) \) then \( f(f(x) \ast f(a)) \ast f(x) = f(a) \). That is \( f(x) \in M(Y) \). Now if \( y \in C(x) \) then \( 0 \ast y = x \) and \( f(0 \ast y) = f(x) \), then \( f(0) \ast f(y) = f(x) \); \( 0 \ast f(y) = f(x) \). That is \( f(y) \in C(f(x)) \).

**Definition (6.3):** Let \( X \) be an RG – algebra, for a fixed \( a \in X \) the map \( R_a : X \rightarrow X \) defined by \( f_a(x) = x \ast a; x \in X \) is called a right map on \( X \). The set of all right maps on \( X \) is denoted by \( R(X) \). The left map on \( X \) is defined dually and is denoted by \( L_a(x) = a \ast x, x \in X \). The set of all left maps on \( X \) is denoted by \( L(X) \).

**Definition (6.4):** A right map is called an idempotent if \( R_o R_a = R_a \) that is \((x \ast a) \ast a = (x \ast a)\), dually the left map.
Lemma (6.5): The left and right maps on an RG – algebra are homomorphism if and only if \( a = 0 \).

**Definition (6.6):** Let \( X \) be an RG–algebra. For a fixed \( a \in X \) the map \( R'_a : X \mapsto X; x \mapsto (x * a) * a \) is called a weak right self–map while \( L'_a : X \mapsto X; x \mapsto a * (a * x) \) is called a weak left self – map. The last map is identical to the identity map on \( X, \forall a \in X \).

**Proposition (6.7):** For any RG – algebra \( X \) we have

i) \( L_a^2 \) is the identity map on \( X \).

ii) Every weak left self map is an idempotent.

iii) \( R_a^2 = R'_a \).

**Proof:** Straight forward.

**Proposition (6.8):** Let \((X; *, 0)\) be an RG – algebra then the following hold:

i) \( R_a(x) * R_a(y) = L_a(y) * L_a(x) = x * y \).

ii) \( L_a(x) * L_a(y) = R_a(y) * R_a(x) = x * y \).

(iii) \((L_a \circ R_a)(x) = L_a(x) * R_a(0) = L_a(x) \quad \text{if} \quad a \in M(X) \)

(iv) \((R_a \circ L_a)(x) = L_a(x) \quad \text{if} \quad a, x \in M(X) \)

(v) \((L_a \circ R'_a)(x) = L_a(x) \quad \text{if} \quad a \in M(X) \)

(vi) \((R'_a \circ L_a)(x) = L_a(x) \quad \text{if} \quad a \in M(x) \quad \text{or} \quad x \in M(X) \)

**Example (5.9):** Let \( X = \{0, a, b\} \) and \((X; *, 0)\) be the algebra given by the table.

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X; *, 0)\) is an RG – algebra and \( M(X) = \{0\}, C(a) = \{b\}, C(b) = \{a\} \).

In example (3.13) we have \( M(X) = \{0,3\}, C(0) = \{0\}, C(1) = \{5\}, C(2) = \{4\}, C(3) = \{3\}, C(4) = \{2\}, C(5) = \{1\} \).

**References**


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