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Combinatorics on Bruhat Graphs and Kazhdan-Lusztig Polynomials

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Abstract

We establish three new combinatorial results on a relation between irregularity of Bruhat graphs and Kazhdan-Lusztig polynomials for Bruhat intervals in all crystallographic Coxeter systems. For our discussion, Deodhar's inequality and R-polynomials play a key role.

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1 Introduction

Kazhdan-Lusztig polynomials and Bruhat graphs

David Kazhdan and George Lusztig discovered some family of polynomials (now known as *Kazhdan-Lusztig (KL) polynomials*) in 1979 [10] in the course of studying representations of Hecke algebras and Coxeter groups. Since then, this family of polynomials have been of great importance in many areas of modern mathematics such as geometry of Schubert varieties and combinatorics of Coxeter groups. In particular, *Bruhat graphs*, introduced by Dyer [6], play an important role; this graph encodes much information on *Bruhat order*, a graded partial order defined on every Coxeter group.

Deodhar's inequality and R-polynomials

After their paper, there have been significant developments in the 1990s. Many researchers, for example, Billey [1], Carrell-Peterson [5], Dyer [7], Kumar [12] and Polo [13], contributed to establishing *Deodhar's inequality*. The Bruhat graph is indeed a powerful tool to state this inequality; see Section 6 for details. For a criterion of an edge relation of such graphs, the *R-polynomials* (which Kazhdan and Lusztig also discovered at the same time in 1979) are convenient.

Outline

The main purpose of this article is to establish three new results on a relation between irregularity of Bruhat graphs and Kazhdan-Lusztig polynomials for Bruhat intervals in all crystallographic Coxeter systems. As mentioned above, Deodhar's inequality and R-polynomials will play a supporting role for our analysis.

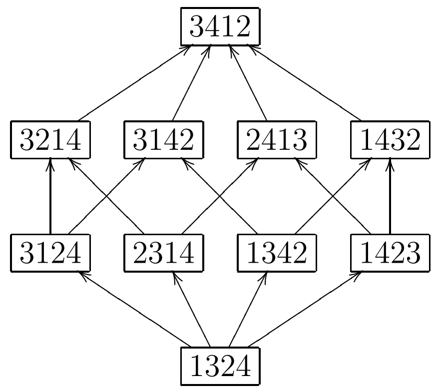
Section 2 begins with basic definitions and notations on Coxeter groups and Bruhat order. Sections 3, 4, 5 and 6 provide more specified definitions and terminology such as Bruhat graphs, R-polynomials, Kazhdan-Lusztig polynomials, Deodhar's inequality and rational singularities of a Bruhat interval. In Section 7, we show the main results as Theorem 7.6, 7.8 and 7.11. With several lemmas, we give proofs. Finally, in Section 8, we close the article with recording some problems for our future work.

2 Notation

We follow common notation in the context of Coxeter groups as books Björner-Brenti [3] and Humphreys [8]. By (W, S, ℓ) (or simply W) we mean a Coxeter system with the set of simple reflections S and the length function ℓ. Unless otherwise specified, u, v, w, x are elements of W and e is the unit. Let $T = \cup_{w \in W} w^{-1}Sw$ denote the set of reflections. Write $u \to w$ if $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$. Define *Bruhat order* $u \le w$ if there exist $v_0, v_1, \dots, v_n \in W$ such that $u = v_0 \to v_1 \to \cdots \to v_n = w$. Let $[u, w] \overset{\text{def}}{=} \{v \in W \mid u \le v \le w\}$ denote a *Bruhat interval*. Often, $\ell(u, w) \overset{\text{def}}{=} \ell(w) - \ell(u)$ abbreviates the length of a Bruhat interval. *Left* and *right descents* of w are $D_L(w) = \{s \in S \mid \ell(sw) < \ell(w)\}$ and $D_R(w) = D_L(w^{-1})$. The *left weak order* is the transitive closure of a relation "$u \to v = su$ for some $s \in S$". Define the *right weak order* replacing su by us above.

Remark 2.1. Furthermore, we assume that W is *crystallographic*; to be precise, for all distinct simple reflections r and s, we have $(rs)^{m(r,s)} = e$ where $m(r, s) \in \{2, 3, 4, 6, \infty\}$. This assumption is to ensure the validity of Facts 5.2 and 6.4.

Figure 1: an example of the Bruhat graph for $[1324, 3412]$



3 Bruhat graphs

Let us begin with a definition of Bruhat graphs, one of our main ideas. Recall that $u \to w$ means $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$.

Definition 3.1. The *Bruhat graph* of W is a directed graph with vertices $w \in W$ and edges $u \to w$.

Also, we will often consider the induced subgraph for a Bruhat interval in W.

Example 3.2. When W is of type A_n (= the symmetric group S_{n+1}), elements of W are permutations and ℓ is the number of inversions; $u \to w$ in W if and only if $w = ut_{ij}$ for some transposition t_{ij} $(i < j)$ and $w(i) > w(j)$. Figure 1 shows the Bruhat graph for $[1324, 3412]$ in $W = \mathrm{A}_3$.

4 R-polynomials

Following [3, Section 5.1], we introduce R-polynomials; this is necessary to introduce Kazhdan-Lusztig polynomials later.

Definition 4.1. Define the *R-polynomials* for W to be a unique family of polynomials $\{R_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q]$ such that

(1) $R_{uw}(q) = 0$ if $u \not\leq w$,

(2) $R_{uw}(q) = 1$ if $u = w$ and

(3) if $s \in S$ and $ws \to w$, then

$$R_{uw}(q) = \begin{cases} R_{us,ws}(q) & \text{if } us \to u, \\ qR_{us,ws}(q) + (q-1)R_{u,ws}(q) & \text{if } u \to us. \end{cases}$$

As mentioned in Introduction, R-polynomials determine an edge relation of Bruhat graphs in the following sense:

Fact 4.2. [3, Exercise 35, Chapter 5] For all u and w, we have

$$\frac{d}{dq} R_{uw}(q)\Big|_{q=1} = \begin{cases} 1 & \text{if } u \to w, \\ 0 & \text{otherwise.} \end{cases}$$

5 Kazhdan-Lusztig polynomials

Following [3, Section 5.1] again, we now introduce Kazhdan-Lusztig polynomials:

Definition 5.1. Define the *Kazhdan-Lusztig polynomials* for W to be a unique family of polynomials $\{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q]$ such that

(1) $P_{uw}(q) = 0$ if $u \not\leq w$,

(2) $P_{uw}(q) = 1$ if $u = w$,

(3) $\deg P_{uw}(q) \leq (\ell(u, w) - 1)/2$ if $u < w$,

(4) if $u \leq w$, then

$$q^{\ell(u,w)} P_{uw}(q^{-1}) = \sum_{v \in [u,w]} R_{uv}(q) P_{vw}(q) \quad \text{and}$$

(5) $P_{uw}(0) = 1$ if $u \leq w$.

Although we just introduced $\{P_{uw}(q)\}$ as polynomials with possibly negative integer coefficients, the following holds under the assumption W to be crystallographic:

Fact 5.2 (Nonnegativity [9]). All coefficients of KL polynomials in W are nonnegative.

Proposition 5.3. $u \leq w \iff P_{uw}(1) \in \{1, 2, 3, \dots\}$.

Proof. By the nonnegativity, we can write $P_{uw}(q) = a_0 + a_1 q + \cdots + a_d q^d$ with a_i all nonnegative integers. Thus, $P_{uw}(1) = a_0 + a_1 + \cdots + a_d$ is a nonnegative integer. If $u \leq w$, then $P_{uw}(0) = a_0 = 1$ so that $P_{uw}(1)$ is a positive integer. If $u \not\leq w$, then $0 = P_{uw}(q) = P_{uw}(1)$. $\square$

Therefore, our analysis will focus on positive integers $\{P_{uw}(1) \mid u \leq w\}$ with a fixed element w. For convenience, we adopt notation $X(w) \overset{\text{def}}{=} [e, w]$. Let us observe further properties of these polynomials.

Fact 5.4 (Invariance [10]). Let $u \in X(w)$ and $r, s \in S$. If $r \in D_L(w)$ and $s \in D_R(w)$, then $P_{ru,w}(q) = P_{uw}(q) = P_{us,w}(q)$. As an easy consequence, we also have $P_{ru,w}(1) = P_{uw}(1) = P_{us,w}(1)$.

Fact 5.5 (Monotonicity, a consequence of [4, Corollary 3.7]). Let $P_{uw}(q) = b_0 + b_1 q + \cdots + b_d q^d$ and $P_{vw}(q) = a_0 + a_1 q + \cdots + a_d q^d$. If $u \leq v \leq w$, then $b_i \geq a_i$ for all i. Consequently, $P_{uw}(1) \geq P_{vw}(1)$.

Fact 5.6 (Existence of a strict inequality [11, Theorem 8.2]). If $P_{uw}(1) > 1$, then there exists an edge $u \to v$ such that

$$P_{uw}(1) > P_{vw}(1) \geq 1.$$

6 Deodhar's inequality and Rational singularities

In this section, we recall *Deodhar's inequality* and a definition of rational singularities. First, we introduce some notations.

Definition 6.1. Let $u \in X(w)$. Set

$$N(u, w) := \{v \in [u, w] \mid u \to v\} \text{ and } \overline{\ell}(u, w) := |N(u, w)|.$$

That is, $N(u, w)$ is the graph-theoretic neighborhood of the bottom vertex u of the Bruhat graph for the Bruhat interval $[u, w]$; $\overline{\ell}(u, w)$ is the number of outgoing edges from u.

Definition 6.2. The *defect* of $[u, w]$ is $\overline{\ell}(u, w) - \ell(u, w)$. We denote this integer by $\mathrm{df}(u, w)$.

We took a little unfamiliar term "defect" from [3, p.168, Exercise 35 (c)]; unfortunately, there seems no common name for this integer in spite of its importance.

Example 6.3. Consider the Bruhat graph back in Figure 1. Let u be the bottom element, v an atom and w the top. Observe that $\mathrm{df}(v, w) = 2 - 2 = 0$ while $\mathrm{df}(u, w) = 4 - 3 = 1$.

As this example suggests, nonnegativity of defects is guaranteed for all Bruhat intervals:

Fact 6.4 (Deodhar's inequality [7]). If $u \leq w$, then $\mathrm{df}(u, w) \geq 0$.

It follows that whenever an integer $\mathrm{df}(u, w)$ is nonzero, then it must be positive. What is the relation between these nonnegative integers $\{\mathrm{df}(u, w) \mid u \in X(w)\}$ and $\{P_{uw}(1) \mid u \in X(w)\}$? Carrell-Peterson's equivalence answers this question:

Fact 6.5. [5, Theorem C] Let $u \in X(w)$. Then the following are equivalent:

(1) $P_{uw}(1) > 1$.

(2) $P_{vw}(1) > 1$ for some $v \in [u, w]$.

(3) $\mathrm{df}(v, w) > 0$ for some $v \in [u, w]$.

Definition 6.6. When any of the conditions above hold, we say that $[u, w]$ is *rationally singular*.

We borrowed this term "rationally singular" from the theory of Schubert varieties; See Billey-Lakshmibai [2] for more details.

7 Main results

In this section, we prove Theorems 7.6, 7.8 and 7.11 as the main results of this article.

Invariance of defects

Let us consider functions $P_{-,w}(1)$ and $\mathrm{df}(-, w)$ on $X(w)$ with a fixed element $w \in W$. As shown in the previous sections, these functions satisfy two weak inequalities: $P_{-,w}(1) \geq 1$ and $\mathrm{df}(-, w) \geq 0$. Moreover, these weak inequalities must be strict simultaneously (Fact 6.5). Under these observations, it is natural to ask:

Question 7.1. Is there any other common property which $P_{-,w}(1)$ and $\mathrm{df}(-, w)$ satisfiy?

Theorem 7.6 gives an affirmative answer to this question. Before that, we need Lemmas 7.2–7.5. The first two lemmas describe relations between a reflection t and a covering edge $u \to v$ in the right weak order.

Notation in this section: let $u, v, w, x \in W$, $s \in S$, $t \in T$ and $T_L(w) = \{t \in T \mid tw \to w\}$.

Lemma 7.2. [3, Proposition 3.1.3] Suppose $u \to v = us$. Then $T_L(u) \subseteq T_L(v)$. As a result, $v \to tv$ implies $u \to tu$.

Lemma 7.3. Let $u = s_1 s_2 \cdots s_n$ be a reduced word (i.e., $n = \ell(u)$) with all $s_i \in S$. Set $t_i := s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2 s_1$ for each $i = 1, 2, \ldots, n$. Then, t_i's are all distinct and $T_L(u) = \{t_1, \ldots, t_n\}$. Moreover, these two assertions are valid independent of the choice of a reduced word of u. As a consequence, if $u \to v = us$, then $T_L(v) = T_L(u) \cup \{vsv^{-1}\}$ (disjoint).

Proof. For the first three statements, see [3, Sections 1.3 and 1.4]. Let us show the consequence. Observe that $s_1 s_2 \cdots s_n s$ is a reduced word for v. By construction of t_i's, the set $T_L(v)$ coincides with $T_L(u)$ except $s_1 s_2 \cdots s_{n-1} s_n s s_n s_{n-1} \cdots s_2 s_1 = vsv^{-1}$. □

Two more lemmas give convenient criteria for Bruhat order.

Lemma 7.4. [3, Lifting Property, Proposition 2.2.7] Let $u < w$. If $s \in D_R(w) \setminus D_R(u)$, then $us \le w$. If $r \in D_L(w) \setminus D_L(u)$, then $ru \le w$.

Lemma 7.5. Let $x \le w$. If $s \in D_R(w)$, then $xs \le w$. If $r \in D_L(w)$, then $rx \le w$.

Proof. Consider the following two cases: If $s \in D_R(x)$, then $xs \to x \le w$; if $s \notin D_R(x)$, then necessarily $x \ne w$ so that $xs \le w$ holds thanks to Lifting Property. The same is true on left descents. □

In other words, all elements x in $X(w)$ are invariant under the action of any s in $D_R(w)$ on the right; it does not matter whether $s \in D_R(x)$ or not. This is what Fact 5.4 asserts implicitly because nonzeroness of KL polynomials is invariant under such actions.

Theorem 7.6. Suppose $u \le w$. Then, for all $r \in D_L(w)$ and $s \in D_R(w)$, we have

$$\mathrm{df}(ru, w) = \mathrm{df}(u, w) = \mathrm{df}(us, w).$$

In particular, this is exactly same to the invariance of KL polynomials:

$$P_{ru,w}(1) = P_{uw}(1) = P_{us,w}(1).$$

Proof. Let $s \in D_R(w)$, $u \to v = us$ and $t' = vsv^{-1}$. We claim that for each $t \in T \setminus \{t'\}$, we have $u \to tu \le w \iff v \to tv \le w$.
(Proof of claim): ($\Longrightarrow$) Suppose $u \to tu \le w$. Then $t \notin T_L(u)$. Since $T_L(v) = T_L(u) \cup \{t'\}$ (disjoint) by Lemma 7.3 and $t \ne t'$, we must have $t \notin T_L(v)$, i.e., $v \to tv$. Moreover, $tv = t(us) = (tu)s \le w$ since $tu \le w$ and $s \in D_R(w)$ (it does not matter whether $s \in D_R(tu)$ or not) as shown in Lemma 7.5.
($\Longleftarrow$) Conversely, suppose $v \to tv \le w$. Since $u \to v = us$ and $v \to tv$, we have $u \to tu$ (Lemma 7.2). Furthermore, $tu \le w$ since $tu = t(vs) = (tv)s \le w$ for the same reason. We proved the claim. ■

Thus, we constructed the bijection $(N(u,w) \setminus \{t'u\}) \cong N(v,w)$. Now taking one more element $v = t'u \in N(u,w)$ into account, we have $\overline{\ell}(u,w) = \overline{\ell}(v,w) + 1$. It follows that

$$\mathrm{df}(u,w) = \overline{\ell}(u,w) - \ell(u,w) = (\overline{\ell}(v,w) + 1) - (\ell(v,w) + 1) = \mathrm{df}(v,w) = \mathrm{df}(us,w).$$

In a similar way, we can show that $\mathrm{df}(ru,w) = \mathrm{df}(u,w)$ for all $r \in D_L(w)$. □

Strict inequality of KL polynomials and defects

The second theorem improves Fact 5.6 on the existence of a strict inequality of $q = 1$ value of KL polynomials.
In the discussion below, for a real polynomial $f(q)$, we denote by $f'(q)$ its derivative.

Lemma 7.7. Let $u \le w$. Then

(1) we have

$$\ell(u,w)P_{uw}(1) - 2P'_{uw}(1) = \sum_{v \in N(u,w)} P_{vw}(1) \quad \text{and}$$

(2) if $[u, w]$ is rationally singular, then $-2P'_{uw}(1) < 0$.

Proof. (1): Differentiate the equation $q^{\ell(u,w)}P_{uw}(q^{-1}) = \sum_{v\in[u,w]} R_{uv}(q)P_{vw}(q)$ in Fact 5.1 (4) once and let $q = 1$. Then the right hand side is $\sum_{v\in N(u,w)} R'_{uv}(1)P_{vw}(1)$ thanks to Fact 4.2.
(2): This follows from the nonnegativity of coefficients (Fact 5.2). □

Theorem 7.8. Suppose $[u, w]$ is rationally singular. Set

$$n := |\{v \in N(u,w) \mid P_{uw}(1) > P_{vw}(1)\}|.$$

Then, we have $n \ge \mathrm{df}(u,w) + 1$.

Proof. Consider $v \in N(u, w)$. By the monotonicity of $P_{-,w}(1)$, we have either $P_{uw}(1) > P_{vw}(1)$ or $P_{uw}(1) = P_{vw}(1)$. Suppose now, toward a contradiction, that $n \le \mathrm{df}(u, w)$. It follows from Lemma 7.7 that

$$\begin{aligned}\ell(u,w)P_{uw}(1) - 2P'_{uw}(1) &= \sum_{v\in N(u,w)} P_{vw}(1)\\ &= \sum_{v:P_{uw}(1)>P_{vw}(1)} P_{vw}(1) + (\overline{\ell}(u,w) - n)P_{uw}(1).\end{aligned}$$

Thus, we obtain the equalities

$$\begin{aligned}\underbrace{-2P'_{uw}(1)}_{<0} &= \sum_{v:P_{uw}(1)>P_{vw}(1)} P_{vw}(1) + (\overline{\ell}(u,w) - n - \ell(u,w))P_{uw}(1)\\ &= \underbrace{\sum_{v:P_{uw}(1)>P_{vw}(1)} P_{vw}(1)}_{\ge 0} + \underbrace{(\mathrm{df}(u,w) - n)P_{uw}(1)}_{\ge 0}\end{aligned}$$

which is a contradiction. □

Maximal rational singularity and defects

Definition 7.9. Suppose $u < w$. Define $\mu(u,w)$ to be the coefficient of $q^{(\ell(u,w)-1)/2}$ in $P_{uw}(q)$.

Since $\deg P_{uw}(q) \leq (\ell(u,w)-1)/2$, $\mu(u,w)$ is the coefficient of possibly the highest degree term of $P_{uw}(q)$. It is not known precisely when $\mu(u,w)$ is nonzero. Theorem 7.11 gives one sufficient condition for $\mu(u,w)$ to be nonzero in terms of the following notion:

Definition 7.10. We say that $[u,w]$ is *maximally rationally singular* if $[u,w]$ is rationally singular and whenever $u < v \leq w$ then $[v,w]$ is not rationally singular.

Theorem 7.11. Suppose $u < w$, $\mathrm{df}(u,w) = 1$ and $[u,w]$ is maximally rationally singular. Then we have $P_{uw}(q) = 1 + q^{(\ell(u,w)-1)/2}$. In particular, $\mu(u,w) = 1 \neq 0$.

Proof. Write $P_{uw}(q) = a_0 + a_1 q + \cdots + a_d q^d$, $a_0 = 1$, $d = (\ell(u,w)-1)/2$ with a_i all nonnegative integers. Now assume that $\mathrm{df}(u,w) = 1$, i.e., $\overline{\ell}(u,w) = \ell(u,w) + 1$ and $[u,w]$ is maximally rationally singular. Use Lemma 7.7 to see

$$\ell(u,w)P_{uw}(1) - 2P'_{uw}(1) = \sum_{v \in N(u,w)} P_{vw}(1).$$

We then obtain

$$\sum_{i=0}^{d} (\ell(u,w) - 2i)a_i = \sum_{v \in N(u,w)} P_{vw}(1) = \sum_{v \in N(u,w)} 1 = \overline{\ell}(u,w) = \ell(u,w) + 1,$$

that is,

$$\ell(u,w) + \sum_{i=1}^{d} (\ell(u,w) - 2i)a_i = \ell(u,w) + 1.$$

Hence

$$\sum_{i=1}^{d} \underbrace{(\ell(u,w) - 2i)a_i}_{\text{nonnegative integer}} = 1$$

(Note: $i \leq d = (\ell(u,w)-1)/2$ implies $\ell(u,w) - 2i \geq 1$). We must have $a_i = 0$ $(1 \leq i \leq d-1)$ and $a_d = 1$ $(= \mu(u,w))$ otherwise $(\ell(u,w) - 2i)a_i$ gives an integer greater than 1. $\square$

8 Future work

In this article, we established three results (Theorems 7.6, 7.8 and 7.11) on a relation between defects of Bruhat graphs and KL polynomials for Bruhat intervals. For subsequence research, we record three problems here.

Problem 8.1. Suppose $u \leq w$.

(1) We showed that for all $r \in D_L(w)$ and $s \in D_R(w)$, we have

$$\mathrm{df}(ru, w) = \mathrm{df}(u, w) = \mathrm{df}(us, w) \text{ and } P_{ru,w}(1) = P_{uw}(1) = P_{us,w}(1).$$

The latter is indeed a consequence of the invariance of polynomials $P_{ru,w}(q) = P_{uw}(q) = P_{us,w}(q)$. From this point of view, it is natural to ask: What should be a q-analog of defects?

(2) We showed that if $P_{uw}(1) > 1$, then there exists at least one edge $u \to v$ such that $P_{uw}(1) > P_{vw}(1) \geq 1$. We wish to see some analogous results for a strict inequality of the function $\mathrm{df}(-, w)$: can we give necessary or sufficient conditions for $\mathrm{df}(u, w) > \mathrm{df}(v, w)$ or $\mathrm{df}(u, w) < \mathrm{df}(v, w)$ to occur?

(3) Consider variants of Theorem 7.11: Suppose $\mathrm{df}(u, w) \geq 2$ and $[u, w]$ is maximally rationally singular. Then, what can we say about $P_{uw}(1)$ and $\mu(u, w)$?

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