Does the Category of Multisets Require a Larger Universe than that of the Category of Sets?

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Abstract

In section 1, the concept of a category is briefly described. In section 2, it is elaborated how the concept of category is naturally intertwined with the existence of a universe of discourse much larger than what is otherwise sufficient for a large part of mathematics. It is also remarked that the extended universe for the category of sets is adequate for the category of multisets as well. In section 3, fundamentals required for adequately describing a category are extended to defining a multiset category, and some of its distinctive features are outlined.

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1. Introduction to categories

The history of science and that of mathematics, in particular, records that at times, a *by-product* may turn out to be of greater significance than the main objective of a research. For example, set theory, which is unarguably regarded as one of the most versatile mathematical theories in modern mathematics, evolved as a by-product while Cantor was engaged in the study of problems relating to trigonometric series. The emergence of the theory of *categories* may be seen as another such an example.

The categories were first introduced by Samuel Eilenberg and Saunders MacLane in 1942-1945, in course of formulating algebraic topology. Not very surprisingly, due to a high level of abstraction involved in the description of a category, some sort of epigrammatic observations like *general abstract nonsense* [Steenrod] appeared. A historically minded reader would recall that similar reaction had appeared at the time set theory or non-Euclidean geometry had appeared. Later, Eilenberg and MacLane wrote that their main objective was to comprehend *natural transformations* which required the notion of *functors* and the latter required *categories* to be defined. In this respect, the notion of categories can be called a *hybrid* by-product. However, this elaboration by the originators should not be construed as a justification for *ultra-abstract* characteristics intrinsic to categories. In fact, categories were born out of necessity while Eilenberg and MacLane were engaged in developing an axiomatic foundation of the relation between mathematical *structures* and the processes preserving them, a fundamental idea first realized and elaborated by Emmy Noether (one of the MacLane's teachers) in 1930s in Poland.

Eilenberg and MacLane define *category C* as an aggregate *Ob* of abstract elements, called the *objects* of *C* and abstract elements *Map*, called *mappings* of the category. The term *Map* is characterized as follows:

(C1) Given three mappings $\alpha_1, \alpha_2, \alpha_3$, the *triple product* $\alpha_3(\alpha_2\alpha_1)$ is defined if and only if $(\alpha_3\alpha_2)\alpha_1$ is defined. That is, whenever either is defined, the associative law $\alpha_3(\alpha_2\alpha_1) = (\alpha_3\alpha_2)\alpha_1$ holds. This triple product is often written as $\alpha_3\alpha_2\alpha_1$.

(C2) The triple product $\alpha_3\alpha_2\alpha_1$ is defined whenever both the products $\alpha_3\alpha_2$ and $\alpha_2\alpha_1$ are defined.

(C3) For each mapping α , there is at least one identity ℓ_1 such that $\alpha\ell_1$ is defined and at least one identity ℓ_2 such that $\ell_2\alpha$ is defined.

(C4) The mapping ℓ_x corresponding to each object *X* is an identity.

(C5) For each identity ℓ there is a unique object *X* of *C* such that $\ell_x = \ell$.

It is remarked that objects play a secondary role and could be entirely done away from the definition; however, it would make the manipulation of the applications less convenient. In fact the definition of category formulated by Eilenberg and MacLane emerged as a helping tool to provide an explicit and rigorous formulation of the notions of functors and natural transformations[1].

Initially, it was not explicit whether the theory of category would turn out to be much more than just an extension of the theory of sets which indeed was the case for about ten years. This perspective altogether changed when categories started getting used in *homology theory* and *homological algebra*. In this connection, MacLane and Grothendieck independently described categories in which the collections of morphisms between two fixed objects were shown to possess an additional structure: given two objects X and Y of a category C , the set $\text{Hom}(X,Y)$ of morphisms form an abelian group.

In the 1960s, Lambek proposed to describe categories as deductive systems. He begins with the notion of a graph consisting of two classes, Arrows and Objects, and two mappings between them, $s: \text{Arrows} \rightarrow \text{Objects}$ and $t: \text{Arrows} \rightarrow \text{Objects}$, namely the source and the target mappings. The arrows are usually called the oriented edge and the objects nodes or vertices. Indeed, along with its rapid growth as a branch of mathematics, after the appearance of Lawvere's Ph.D. thesis in universal algebra, it started getting used in computer Science. In view of the fact that a computer is not good at viewing concrete diagrams, category theory is being extensively used in computer science mainly because it offers a constructive mathematical structure to describe an object.

Soon after, category theory became an autonomous field of research that has now occupied a central position in most of the branches of mathematics, some areas of theoretical computer science where they correspond to *types* (a data type is a set of data with values having predefined characteristics like integer or floating point, usually a limited number of such data are built into a language and this corresponds to a category), and mathematical physics where categories are used to describe *vector spaces* [2]. In particular, as category theory basically studies structures in terms of the mappings between them, many branches of modern mathematics could be conveniently described in terms of categories; for example, category of sets, category of relations, category of groups, etc., and most importantly, doing so often reveals deep insights and similarities between seemingly different areas of mathematics (see [3],[4], for details).

The notion of category generalizes those of a *preorder* and *monoid* and as well provides unification within set theoretical environment, thereby organizing and unifying much of mathematics. By now it has emerged as a powerful language or a conceptual framework providing tools to characterize the universal components of a family of structures of a given kind and their relationships. The study of categories consists in axiomatically comprehending what is characteristically common in various classes of related

mathematical structures. This objective is attained by exploiting *structure-preserving functions* between them. It brings to light, makes explicit, and abstracts out the relevant structures often hidden in following traditional approaches. It also looks for the universal properties holding in the categories of structures one is working with (see [5], for details).

Essentially, a category is an algebraic structure consisting of a collection of *objects*, linked together by a collection of *arrows (morphisms)* that have two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. Objects and arrows are abstract entities of any kind. The type of objects depends upon the chosen mathematical structure. For example, in set theory, the objects are sets; in group theory, the objects are groups; and in topology, the objects are topological spaces.

Morphism is the central concept in a category and that, a category C consists of the collection $\text{Mor}C$ of the morphisms of C . Objects of C are associated with identity morphisms 1_A , since 1_A is unique in each set $\text{Mor}C(A, A)$ and uniquely identifies the object A .

Morphisms are structure-preserving maps. Morphism is an *arrow* linking an object, called the *domain*, to another object, called the *codomain*. The notion of morphism connotes differently depending on the type of spaces chosen. In set theory, morphisms are functions; in group theory, they are group homomorphisms; and in topology, they are continuous functions; etc.

A morphism has two parts: the domain (or source) and the codomain (or target). If a morphism f has domain A and codomain B , we write $f: A \rightarrow B$ and say that f is a *morphism from A to B* . Thus, morphism is represented by an arrow from its domain to its codomain. The collection *hom-Class* of all morphisms from A to B is denoted $\text{hom}(A, B)$ or $\text{Hom}(A, B)$ or $\text{hom}_C(A, B)$ or $\text{Mor}(A, B)$ or $C(A, B)$.

A morphism $f: A \rightarrow B$ in a category C is called

- (i) a *monomorphism* if it is left cancellable i.e., for every pair of morphisms $g, h: C \rightarrow A$ if $f \circ g = f \circ h \Rightarrow g = h$;
- (ii) a *split monomorphism (or section or coretraction)* if it is left invertible i.e., there exists a morphism $g: B \rightarrow A$ such that $g \circ f = 1_A$;
- (iii) an *epimorphism* if it is right cancellable i.e., for every pair of morphisms $g, h: B \rightarrow C$, $g \circ f = h \circ f \Rightarrow g = h$;
- (iv) a *split epimorphism (or retraction)* if it is right invertible i.e., there exists a morphism $g: B \rightarrow A$ such that $f \circ g = 1_B$;
- (v) a *bimorphism* if it is both a monomorphism and an epimorphism; and
- (vi) an *isomorphism* if it is both split monomorphism and split epimorphism i.e., if it is invertible (see [6], for details).

Schematically, a category C consists of the following:

- (i) A Class $\text{Ob}(\mathbf{C})$ of *objects*.
- (ii) A class $\text{hom}(\mathbf{C})$ of *morphisms or maps or arrows*.
- (iii) A binary operation called *composition of morphisms*, such that for any three objects A, B and C , we have $\text{hom}(B, C) \times \text{hom}(A, B) \rightarrow \text{hom}(A, C)$. That is, the composite of $f: A \rightarrow B$ and $g: B \rightarrow C$, written $g \circ f: A \rightarrow C$ or $gf: A \rightarrow C$, is an operation governed by the following two axioms viz.,
 - (a) Associativity: if $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
 - (b) Identity: For every object X , there exists a morphism $1_X: X \rightarrow X$ called the *identity* morphism for X , such that for every morphism $f: A \rightarrow B$, we have $1_B \circ f = f = f \circ 1_A$.

2. A characterization of the *Universe* for Categories and multiset categories

The universality of a categorical structure lies in the fact that instead of investigating a given individual mathematical structure, it endeavors to unfold the universal structural properties common to various classes of related mathematical structures. This goal of defining the kernel of the classes of related mathematical structures is achieved by characterizing appropriate structure-preserving functions between them. The powerful language of category theory legitimately admits speaking of categories of all *sets* and *set functions*, categories of all *multisets* and *multiset functions*, categories of all *groups* and *group homomorphisms*, etc., which have no place in ZFC. The notion of *big sets* must underline whatever foundation is endorsed for category. The arbitrarily restricted language of ZFC is found insufficient to deal with categories and all fields of mathematics using categorical foundations, like algebraic geometry, etc., which require foundations providing means to legitimately perform those constructions and operations that are mathematically interesting and useful (Herrlich and Strecker [7], p.331). Let us ponder for a while on how such arbitrarily large sets could be admitted in set theory-like languages without leading to paradoxes. It may be recalled that in ZFC, the axiom of infinity, either in Zermelo form:

$$(\exists y) [\phi \in y \wedge \forall x (x \in y \rightarrow \{x\} \in y)],$$

or in von Neumann form:

$$(\exists y) [\phi \in y \wedge \forall x (x \in y \rightarrow x \cup \{x\} \in y)],$$

was introduced to grant the existence of infinite sets. However, it is incapacious to provide the existence of *big sets* as required in category theory.

In order to enlarge the restricted framework of ZFC; von Neumann, in his paper *An axiomatization of set theory* [8], endeavored to boldly admit the collections that were considered *inconsistent manifolds* [Cantor] or *inconceivable entities* [Russell]. von Neumann's system, due to using function rather than sets as its primitive term was found a bit cumbersome to easily follow. Efforts of Paul Bernays, Kurt Gödel, R. M. Robinson and others provided it a more comprehensive form, which came to be known

by the acronym NBG. In NBG, *classes* (not *sets*) of all objects (sets) satisfying a given property exist. It admits classes (not the sets) and $\in$ -relation as primitives. Gödel [9] explicitly introduces *set* also as a primitive. A class is a set if it is contained in some other class. A class is called *proper* if it is not a set. NBG admits a class form of comprehension principle:

$$\exists X [\forall x (x \in X \rightarrow A(x))],$$

where x is a set variable, X is a class variable and $A(x)$ contains no bound class variables. The class-form of the comprehension principle grants the existence of classes like the class of all ordinals or that of the *universal class* $\{x: x = x\}$, etc., as legitimate mathematical objects. However, there is no class of all classes in NBG. It is still short of providing adequate foundation for categories which will require the existence of *big sets* (Dalen [10] and Mendelson [11] are excellent expositions on NBG).

It was the great insight of a 20th century legendary figure in the mathematical world Alexander Grothendieck (1928-) that a highly abstract concept of *universal properties* could be coherently characterized that made big sets mathematically treatable and provided there by an adequate grammar for the language of category theory. As a matter of fact, the *universality* aspect of categories was *coded* by a set of properties characterizing a *universe* known as *Grothendieck universe* or simply a *universe* (see Artinet *al* [12], for various details). We briefly describe the properties defining the *universe* (Schubert [13] contains details) as follows:

A nonempty set $\mathfrak{U}$ is said to be a *universe*, denoted $Univ \mathfrak{U}$, if and only if

$$(u1) \forall x [x \in \mathfrak{U} \supset x \subseteq \mathfrak{U}];$$

$$(u2) \forall x \forall y [(x \in \mathfrak{U} \wedge y \in \mathfrak{U}) \supset \{x, y\} \in \mathfrak{U}];$$

$$(u3) \forall x [x \in \mathfrak{U} \supset p(x) \in \mathfrak{U}]; \text{ and}$$

$$(u4) \forall x \forall y [(x \in \mathfrak{U} \wedge y \in \mathfrak{U}) \supset \bigcup_{i \in x} y_i \in \mathfrak{U}];$$

i.e., for any family $Y = \{y_i\}_{i \in x}$ of elements of $\mathfrak{U}$, with $x \in \mathfrak{U}$ as the index set, the union $\bigcup_{i \in x} y_i \in \mathfrak{U}$. Thus for a given $Univ \mathfrak{U}$, the elements of $\mathfrak{U}$ can be comprehended as sets, subsets of $\mathfrak{U}$ as classes and all other sets as *unimaginably massive* (Murfet [14] contains some further details). Any set $x \in \mathfrak{U}$ is called *small*. A set or structure is called *large* if it is at least as large as some universe. A number of results that follow from the set of axioms $\{(u1- u4)\}$ have found applications in many areas of mathematics, including category theory. However it should be noted that from the set of axioms $\{(u1- u4)\}$, $\aleph \subset \mathfrak{U}$ is deducible, but not necessarily $\aleph \in \mathfrak{U}$.

Most significantly, on strength of the aforesaid characterization of the notion of universes, the following *axiom of universe* viz.,

$$A\mathfrak{U}: \forall x \exists \mathfrak{U} [Univ \mathfrak{U} \wedge x \in \mathfrak{U}]$$

is introduced. In other words, $A\mathfrak{U}$ states that *every set is contained in some universe* which, in turn, grants legitimacy to speaking of *set of all sets in a given universe*. In particular, the axiom of universes guarantees the existence of at least one infinite universe with the potential to contain $\aleph_0$, the set of natural numbers (McLarty [15] is an excellent exposition on various foundational aspects of universes).

It is observed in Drake and Singh [16], P. 190; that the *axiom of universe* is very close to the full second order version of *cumulative type* structure and, in turn, to Fraenkel's strong axiom of infinity (Levy [17]; Singh and Singh [18]). In Kruse [19], it is pointed out that the notions of *Grothendieck universes* and *small super complete near model* are equivalent.

Summarily, *ZFC plus $A\mathfrak{U}$* provides adequate means to construct structures requiring *big sets* such as, the set of all categories of sets or multisets, etc. In view of the fact that multiset theory is a conservative extension of the theory of sets [20], it is in order to define a category of multisets in the extended universe developed above.

3. Multiset Category

3.1 Multisets and multiset functions

Let us recall that a standard (or ordinary) set is a welldefined collection of distinct elements. In course of time, specially in order that set theoretic tools could be made applicable to solve problems which would require generalizing some too restrictive assumptions used to define ordinary sets, various *nonstandard set theories* have appeared. In particular, if multiple but finite occurrences of any element of a set are allowed, we get a generalization of the notion of set, called a *Multiset* (*mset*, for short). Besides in mathematics, as multisets are being extensively used in theoretical computer science, biosystem (membrane computing, DNA computing, etc.), economics, social sciences, etc., categorical model of multisets are likely to play a significant role, both in hard and soft sciences ([20] and [21] are excellent expositions on applications of multisets).

As suggested in [22], collections admitting objects with finite multiplicities could be viewed in two different ways: Sets with distinguishable repeated elements (e.g., people or vehicles sharing a common property), called *multisets*; and sets with indistinguishable repeated elements (e.g., *soup* of elementary particles), called *multinumerals*; however, in most of the literature of multisets, this terminology is usually reversed. Syropoulos[23], in particular, calls the former *Real multiset* and the latter *multiset*. It is important to emphasize that the concepts viz., Real multisets and multisets are associated with a set equipped with an *equivalence relation* or a *function*, respectively. However, taking developments pertaining to admitting multiple occurrences of an object in a system, it is the notion of multiset which has mostly been

exploited. In course of developing category of multisets, we put emphasis on the notion of multisets associated with a set and an equivalence relation in line with [22].

A Multiset is an unordered collection of *objects* in which, unlike a standard (Cantorian) set, duplicates or multiples of objects are admitted. In other words, an mset is a collection in which objects may appear more than once and each individual occurrence of an object is called its *element*. All duplicates of an object in an mset are indistinguishable. The objects of an mset are the distinguishable or distinct elements of the mset. The distinction made between the terms *object* and *element* does enrich the multiset language. However, use of the term *element* alone may suffice if there does not arise any confusion (see [24], for details). The set of all objects of an mset is called its *root set*, annotated A^* .

The use of square brackets to represent an mset is quasi-general. Thus, an mset containing one occurrence of a , two occurrences of b , and three occurrences of c , is annotated $[[a, b, b, c, c, c]]$ or $[a, b, b, c, c, c]$ or $[a, b, c]_{1,2,3}$ or $[a, 2b, 3c]$ or $[a.1, b.2, c.3]$ or $[1/a, 2/b, 3/c]$ or $[a^1, b^2, c^3]$ or $[a^1b^2c^3]$. For convenience, the curly brackets are also used in place of the square brackets. Note that the root set of the aforesaid mset is the set $\{a, b, c\}$.

For various application purposes, we may regard a multiset $[a, a, b]$ as being really of the form $[a, a', b]$ where a and a' are different objects of the same sort and b is of different sort from that of a and a' . In this regard, when elements of multisets are considered, elements of distinct sorts will generally be denoted by distinct letters and elements of the same sort will be denoted by the same letter with dashes distinguishing different elements of that sort (see [22], for details).

Formally, a multiset A is a pair (A_0, ρ) , where A_0 is a set and ρ an equivalence relation on A_0 . The set A_0 is called *the field of the multiset*. Elements of A_0 in the same equivalence class will be said to be of the same sort, elements in different equivalence classes will be said to be of different sorts. For example, an mset $[a, a, b, c, c, c, d]$ is the multiset $[a, a', b, c, c', c'', d]$, where all are seen to behave as different objects; but a, a' are of the same sort and c, c', c'' are also of the same sort, while b, d are of different sorts from the others. In other words, various equivalence classes determine the sorts. Whenever a multiset A is mentioned, its field will usually be denoted A_0 . The pair (A_0, ϕ) , where A_0 is a set and ϕ the empty relation on A_0 , is actually an ordinary set.

Multiset Functions

Let $A = (A_0, \rho)$ and $B = (B_0, \sigma)$ be multisets. A multiset function from A to B , written as $f: A \rightarrow B$, is a function $f: A_0 \rightarrow B_0$ which respects sorts i.e., if $a, a' \in A_0$ and $a \rho a'$, then $f(a) \sigma f(a')$. For example, $f: [a, b, c] \rightarrow [d, d', e]$, defined by $f(a) = d'$, $f(b) = e$, $f(c) = d$, and $g: [a, a', b] \rightarrow [c, c', d, e, f]$, defined by $g(a) = c$, $g(b) = d$, $g(a') = c'$, are multiset functions, while $h: [a, a', b] \rightarrow [c, c', d]$, defined by $h(a) = c'$, $h(a') = d$ and $h(b) = c$, is not an mset function since $a, a' \in A_0$ and $a \rho a'$ but $h(a) \not\sigma h(a')$.

Empty mset function

Let $\theta = (\phi_0, \phi)$, where ϕ_0 is an empty set and ϕ an empty relation on ϕ_0 . Then, θ is said to be an *empty* multiset. That is, an empty set together with an empty equivalence relation defined on it determined an empty multiset.

Identity mset function

Let $A = (A_0, \rho)$ be an mset. Then a function $I: A \rightarrow A$ is an mset function $I: A_0 \rightarrow A_0$ called an *identity mset* function on A , denoted I_A , satisfying $I_A(a) = a, \forall a \in A_0$.

Composition of mset function

Let $A = (A_0, \rho)$, $B = (B_0, \sigma)$, and $C = (C_0, \tau)$ be multisets and $f: A \rightarrow B$, $g: B \rightarrow C$ be multiset functions, then the *composition* or *composite* or *relative product* of f and g , denoted gof (or fog , depending on the convention one follows) is the mset function $\text{gof}: A \rightarrow C$ which respects sorts (i.e., if $a, a' \in A_0$ and $a \rho a'$, then $\text{gof}(a) \tau \text{gof}(a')$), satisfying $\text{gof}(a) = g(f(a)), \forall a \in A_0$. Sometimes, if gof is defined, it is called the *left-composition* of g with f or, g is called *left-composable* with f . Moreover, on the same lines, for mset functions $f: A \rightarrow A$, and $g: A \rightarrow A$, the composite mset functions fog , gof , fof , gog , etc., can be computed. The composition of mset functions is associative.

In order to have a better comprehension of the concept of multiset category, we briefly outline below that the *equivalence relation approach* to define multiset and multiset functions is equivalent to the usual *functional approach*.

Akin to defining a set function, the underlying assumption for defining an mset function between two multisets is not to allow mapping of indistinguishable elements to distinguishable elements; however, images of distinct elements of the domain need not be distinct elements of the range. In turn, it amounts to defining a function between the *root sets* of the two multisets involved along with some consistency conditions observed.

Let A and B be two multisets. An mset relation f from A to B , sometimes denoted $f: A \rightarrow B$ is said to be an mset function if for every element $m/x \in \text{Dom } f$ there exists exactly one element $n/y \in \text{ran } f$ such that $(m/x, n/y) \in f$, satisfying the consistency condition that the pair (x, y) occurs only $C_1(x, y)$ times. The symbol $C_i(x_1, x_2, \dots, x_n)$, $i = 1, 2, \dots, n$, denotes the count of the i^{th} coordinate in the n -tuple $(m_1/x_1, m_2/x_2, \dots, m_n/x_n)/k$, where k denote the number of times the tuple $(x_1, x_2, \dots, x_n)$ occurs.

For example, let $A = \{3/x, 5/y\}$ and $B = \{2/p, 6/r\}$ be two mset. Then, $f = \{(3/x, 6/r)/3, (5/y, 2/p)/5\}$ is an mset function from A to B .

The mset function $f: A \rightarrow B$ is an injection (one-one) if

- (i) $f: A^* \rightarrow B^*$ is an injection, and
- (ii) $\forall z(z \in A^* \Rightarrow m_A(z) \leq m_B(f(z)))$.

Thus, distinct elements of the domain under an injective mset functions are mapped into distinct elements of the range. That is, $m_1/x_1 \neq m_2/x_2 \in \text{dom } f \Rightarrow f(m_1/x_1) \neq f(m_2/x_2) \in \text{ran } f$.

In other words, f is one-one if no two elements in $\text{dom } f$ have the same image under f with $C_1(x, y) \leq C_2(x, y)$ for all $(x, y) \in f$.

For example, $f: \{3/x\} \rightarrow \{10/y\}$ defined by $f = \{(3/x, 10/y)/3\}$ is an injection.

The mset function $f: A \rightarrow B$ is a surjection (onto) if

- (i) $f: A^* \rightarrow B^*$ is a surjection, and
- (ii) $\forall z(z \in A^* \Rightarrow m_A(z) \geq m_B(f(z)))$.

In other words, an mset function f is onto if $\text{ran } f = \text{co-dom } f$ with $C_1(x, y) \geq C_2(x, y)$ for all $(x, y) \in f$. Note that the distinct element of the $\text{dom } f$ need not be mapped under f to distinct elements of the $\text{ran } f$.

For example, $f: \{5/x\} \rightarrow \{4/y\}$ defined by $f = \{(5/x, 4/y)/5\}$ is a surjection.

The mset function $f: A \rightarrow B$ is a bijection (one-one and onto) if

- (i) $f: A^* \rightarrow B^*$ is a bijection, and
- (ii) $\forall z(z \in A^* \Rightarrow m_A(z) = m_B(f(z)))$.

In other words, an mset function f is bijection if f is both one-one and onto with $C_1(x, y) = C_2(x, y)$ for every pair $(x, y) \in f$.

For example, $f: \{3/x, 5/y\} \rightarrow \{5/p, 3/r\}$ defined by $f = \{(3/x, 3/r)/3, (5/y, 5/p)/5\}$ is a bijection. Note also that an mset function can neither be an injection nor a surjection.

For example, $f: \{6/x, 2/y\} \rightarrow \{3/p, 4/r\}$ defined by $f = \{(6/x, 3/p)/6, (2/y, 4/r)/2\}$ is such an mset function.

An mset function on an mset A is said to be an *identity* function on A , denoted I_A or $I: A \rightarrow A$, if $I(m/x) = m/x$ with $C_1(x, x) = C_2(x, x)$.

Let A, B, C be msets. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two mset functions. The *composition* or *composite* or *relative product* of f and g , denoted gof (or fog , depending on the convention one follows) is the mset function defined by $(\text{gof})(x) = g(f(x))$ with $C_1(x, z) = \min \{C_1(x, y), C_1(y, z)\}$ and $C_2(x, z) = \min \{C_2(x, y), C_2(y, z)\}$. Sometimes, if gof is defined, it is called the *left-composition* of g with f or, g is called *left-composable* with f . Moreover, on the same lines, for mset functions $f: A \rightarrow A$, and $g: A \rightarrow A$, the composite mset functions fog , gof , fof , gog , etc., can be computed. The composition of mset functions is associative.

Note that, the underlying assumption for defining composition of functions is $\text{ran } f \subseteq \text{dom } g$; otherwise gof is empty.

Example:

Let $f = \{(3/x, 3/p)/3, (4/y, 5/p)/4, (3/z, 6/q)/3\}$, and $g = \{(5/p, 7/t)/5, (6/q, 8/u)/6\}$. Then $\text{gof} = \{(3/x, 3/t)/3, (4/y, 5/t)/4, (3/z, 6/u)/3\}$.

From above, $\text{dom } f = \{3/x, 4/y, 3/z\}$ and $\text{ran } f = \{5/p, 6/q\}$; $\text{dom } g = \{5/p, 6/q\}$ and $\text{ran } g = \{7/t, 8/u\}$. Since, $\text{ran } f \subseteq \text{dom } g$, gof is well defined.

Moreover, $\text{dom } (\text{gof}) = \{3/x, 4/y, 3/z\}$ and $\text{ran } (\text{gof}) = \{5/t, 6/u\}$. This shows that $\text{dom } (\text{gof}) \subseteq \text{dom } f$ and $\text{ran } (\text{gof}) \subseteq \text{ran } g$ granting the consistency conditions for gof to be defined as a composite relation.

3.2 Multiset Category

Multisets (considered as objects) and multiset functions (considered as morphisms) together determine the category of multisets, denoted **Mul**.

Typically, multisets can themselves be regarded as categories such that if A is a multiset and $a, b \in A$, there is an arrow from a to b if a and b are of the same sort, and no arrows from a to b if a and b are of different sorts.

DEFINITIONS:

Let $A = \langle A_0, \rho \rangle$ and $B = \langle B_0, \sigma \rangle$ be two msets. Then

(i) $f: A \rightarrow B$ is a monomorphism in **Mul** iff $f: A_0 \rightarrow B_0$ is one-to-one, sort-preserving, and $|S_i| \leq |T_k|$ where S_i is a sort in A , T_k be the corresponding sort in B , and $|S_i|$ is the cardinality of S_i , $|T_k|$ is the cardinality of T_k .

For example, $f: [a, a', b] \rightarrow [c, c', d, e, e']$, defined by $f(a) = c$, $f(b) = d$, $f(a') = c'$, is a monomorphism; $f: [a, a', a''] \rightarrow [b, b']$ cannot be a monomorphism.

Note that objects of the same sort cannot be mapped to objects of different sorts.

(ii) $f: A \rightarrow B$ is an epimorphism in **Mul** iff $f: A_0 \rightarrow B_0$ is onto, sort-preserving, and $|S_i| \geq |T_k|$. For example, $f: [a, a', b] \rightarrow [c, d]$, defined by $f(a) = f(a') = d$, $f(b) = c$, is an epimorphism; $f: [a, a'] \rightarrow [b, b', b'']$ cannot be an epimorphism.

(iii) $f: A \rightarrow B$ is an isomorphism in **Mul** iff $f: A_0 \rightarrow B_0$ is bijection, sort-preserving, and $|S_i| = |T_k|$. For example, $f: [a, a', b] \rightarrow [c, c', d]$, defined by $f(a) = c'$, $f(b) = d$, $f(a') = c$, and $g: [a, b] \rightarrow [c, d]$, defined by $g(a) = d$, $g(b) = c$, are isomorphisms. We choose to call the aforesaid conditions required to hold between the cardinalities of the sorts as the *consistency criteria* for the existence of mono, epi, and iso-morphisms in **Mul**. It is distinctive to note that the category **Mul**, unlike that of the category **Set**, is not balanced, since every bimorphism is not an isomorphism [25].

It may be emphasized that the aforesaid consistency criteria have been missing in previous literature dealing with multiset isomorphism.

In this respect, one may envisage that such categories would require a further extension of the *universe* viz; ZFC plus A^Ψ . We note that, as the notion of *power multiset* is reducible to *power set* in multiset theory which is a conservative extension of ZFC [21], ZFC plus A^Ψ is sufficient.

Objects in a Mul

An object I in a category $\mathbf{C}$ is said to be an *initial* object of $\mathbf{C}$ if for every object X of $\mathbf{C}$ there is a unique morphism $I \rightarrow X$. Since for every object X in **Mul** there is a unique morphism $\emptyset \rightarrow X$, the empty multiset $\emptyset$ can be regarded as an initial object in **Mul**.

Also, an object T in a category $\mathbf{C}$ is said to be a *terminal* object of $\mathbf{C}$ if for each object X of $\mathbf{C}$ there is exactly one morphism $X \rightarrow T$. In **Mul**, an object satisfying this universal mapping property is the one-element multiset, denoted **1**. Indeed, for each object X of **Mul**, there is a unique morphism $X \rightarrow \mathbf{1}$. Hence, any singleton mset is a terminal object

in **Mul**. Moreover, since a zero object is an object that is both initial and terminal, there is no zero object in **Mul**.

Thus, similar to the category of sets, the category of multisets has a unique initial object, but many terminal objects which are isomorphic to each other.

Points of an object in a **Mul**

A point of an object X in a category **C** is an arrow $\mathbf{1} \rightarrow X$, where $\mathbf{1}$ is terminal. Hence, in the category **Mul**, the points of an object A are precisely the elements of A . Suppose $A = [a, a', b]$ is an object in the category **Mul**, then A has three points and, each point of A can be seen to *point to* exactly one element of A and every element of A is the value of exactly one point of A . Moreover, if $f: A \rightarrow B$ and a is a point of A , then fa is a point of B . The terminal object in **Mul** has one point. The terminal object $\mathbf{1}$ in **Mul** helps in separating arbitrary arrows. For the parallel arrows $f, g: A \rightarrow B$, if for each point a of A we have $fa = ga$, then $f = g$. Thus, if two arrows *agree* on points, they are the same arrows. For example, given $f: A \rightarrow B$ and $g: A \rightarrow B$, if $fa = ga$ for every point $a: \mathbf{1} \rightarrow A$, then we can conclude that $f = g$. In other words, if $f \neq g$, then there exists at least one point $a: \mathbf{1} \rightarrow A$ for which $fa \neq ga$. Similar to **Set**, two arrows in **Mul** are equal if they have the same domain and codomain and if they have the same value at every element of their domain.

4. Conclusions

After briefly describing the notion of a category, the paper presents a characterization of the *universe* that would grant legitimacy to speaking of *set of all sets and multisets in a given universe*. The equivalence of the usual *functional approach* and the *equivalence relation approach* to define multiset and multiset functions is demonstrated. Further, the definition of multiset isomorphism, given in [22], is strengthened and thereby, it is shown that multisets (considered as *objects*) and multiset functions (considered as *morphisms*) together determine the category of multisets in which the *terminal* and *initial* objects exist but there is no *zero* object.

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