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On the Structure of Some Groups Containing PSL(3,3) wr PSL(3,5)

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Abstract

In this paper, we will generate the wreath product $PSL(3,3)wr\ PSL(3,5)$ using only two permutations. We will show the structure of some groups containing the wreath product $PSL(3,3)wr\ PSL(3,5)$. The structure of the group constructed is determined in terms of wreath product$(PSL(3,3)wrPSL(3,5))wrC_k$. Some related cases are also included. Also, we will show that S_{403k+1} and A_{403k+1} can be generated using the wreath product$(PSL(3,3)wr\ PSL(3,5))wrC_k$ and a transposition in S_{403k+1} and an element of order 3 in A_{403k+1} . We will also show thatS_{403k+1} and A_{403k+1} can be generated using the wreath product $PSL(3,3)wr\ PSL(3,5)$ and an element of order k+1 .

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1 Introduction

Hammas and Al-Amri [1], have shown that A_{2n+1} of degree $2n+1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1} . They also gave the symmetric generating set of Groups $A_{kn+1}$and S_{kn+1} using S_n [5] .

Shafee [2] showed that the groups A_{kn+1} andS_{kn+1} can be generated using the wreath product $A_m wr\ S_a$ and an element of order k+1. Also she showed how to generateS_{kn+1} and A_{kn+1}symmetrically using n elements each of order k+1.

In [3], Shafee showed that the groupsA_{252k+1} and S_{252k+1} can be generated using the wreath product $L_2(13)wr\ L_2(17)$ and an element of order k+1.

The Linear groups $PSL(3,3)$ and $PSL(3,5)$ are two groups of the well known simple groups. In [6], they are fully described. They have presentations in [6] as follows :

$$PSL(3,3) =< X, Y \ \ |X^6 = Y^3 = (XY)^4 = (X^2Y)^4 = (X^3Y)^3 = [X^2, (YX^2Y)^2] = 1 > . \tag{1}$$

Using GAP$PSL(3,3)$ can be generated using two permutations as follows:

$$PSL(3,3) =< (5,8,11)(1,5)(6,9,12)(7,10,13)(1,2,5)(3,8,7)(4,11,6)(9,10,13) > . \tag{2}$$

Using GAP$PSL(3,5)$ can be generated using two permutations as follow:

$$PSL(3,5) =< (\alpha, \beta >,$$

where

$$\alpha = (3,4,5,6)(8,11,10,9)(12,22)(13,26,15,24)(14,,23,16,25)(17,27)(18,31,20,29)(19,28,21,30)$$

$$\beta = (1,12,16,13)(2,9,24,29)(3,21,30,22)(4,28,7,20)(5,25,18,11)(10,31,17,23)$$

Here we will generate the wreath product $PSL(3,3)wr\ PSL(3,5)$ using only

two permutations and we will show the structure of some groups containing the wreath product$PSL(3,3)wr\ PSL(3,5)$. The structure of the groups obtained is determined in terms of wreath product $(PSL(3,3)wr\ PSL(3,5))wrC_k$.

Some related cases are also included. We will show that S_{403k+1} and A_{403k+1} can be generated using the wreath product$(PSL(3,3)wr\ PSL(3,5))wrC_k$ and a transposition in $S_{403k+1}$and an element of order in A_{403k+1} . We will also show thatS_{403k+1} and A_{403k+1}can be generated using the wreath product $PSL(3,3)wr\ PSL(3,5)$ and an element of order .

2 PRELIMINARY RESULTS

DEFINITION 2.1.[6] The general linear group $GL_n\ (q)$consists of all the $n{\times}n$ matrices that have non-zero determinant over the field F_q with q-elements. The special linear group $SL_n(q)$ is the subgroup of $GL_n\ (q)$ which consists of all matrices of determinant one. The projective general linear group $PGL_n(q)$ and projective special general linear group $PSL_n(q)$ are the groups optained from$GL_n\ (q)$ and$SL_n(q)$ .The projective special general linear group$PSL_n(q)$ is also denoted by $L_n(q)$.The orders of these groups are

$\mid GL_n\ (q) \mid= (q-1)N, \mid SL_n(q) \mid=\mid PGL_n(q) \mid= N, \mid PSL_n(q) \mid=\mid L_n(q) \mid= \frac{N}{d}$,where

$N = q^{\frac{1n(n-1)}{2}}(q^n-1)(q^{n-1}-1)...(q^2-1)$ and $d = (q-1, n)$

DEFINITION 2.2.[4] Let A and B be groups of permutations on non empty sets Ω_1 and Ω_2, respectively, where $\Omega_1 \cap \Omega_2 = \phi$. The wreath product of A and B is denote by $AwrB$ and defined as $AwrB = A^{\Omega_2} \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of A and a mapping θ, where $\theta : B \to Aut(A^{\Omega_2})$ is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that

$$|AwrB| = (|A|)^{|\Omega_2|}|B|. \tag{3}$$

THEOREM 2.3 (Jorden-Moore)[7] The group$PSL_n(q)$ is simple if and only if $q \geq 3$.

THEOREM 2.1 [4] *Let G be the group generated by the n-cycle* $(1, 2, \cdots, n)$ *and the 2-cycle* (n, a)*. If* $1 < a < n$, *is an integer with* $n = am$ *, then*

$$G \cong S_m wr C_a. \tag{4}$$

.

THEOREM 2.5 [4] *Let* $1 \leq a \neq b < n$ *be any integers.Let* n *be an ood integer and let* G *the group generated by the n-cycle* $(1, 2, \cdots, n)$ *and the 3-cycle* (n, a, b)*. If* $hcf\ (n, a, b) = 1$, *then* $G \cong A_n$ .While if n can be even then

$$G \cong S_n. \tag{5}$$

THEOREM 2.6[4] Let $1 \leq a \leq n$ be any integer. Let $G = \prec (1, 2, ..., n), (n, a) \succ$. If $hcf(n, a) = 1.$, then $G \cong S_n$.

THEOREM 2.7 [4] *Let* $1 \leq a \neq b < n$ *be any integers.Let* n *be an even integer and let* G *the group generated by the n-cycle* $(1, 2, \cdots, n)$ *and the 3-cycle* (n, a, b). *Then*

$$G \cong A_n. \tag{6}$$

3 THE RESULTS

THEOREM 3.1 *The wreath product* $PSL(3, 3)wr\ PSL(3, 5)$ *can be generated using* two permutations, the first is of order $403k$ *and the second is of order 4.*

Proof: Let $G = \prec X, Y \succ$,Where:$X = (1, 2, 3, ..., 403)$,

$Y = (3, 4, 5, 6)(8, 11, 10, 9)(12, 22)(13, 26, 15, 24)(14, 23, 16, 25)(17, 27)(18, 31, 20, 29)$ $(19, 28, 21, 30)(1, 12, 16, 13)(2, 9, 24, 29)(3, 21, 30, 22)(4, 28, 7, 20)(5, 25, 18, 11)$ $(10, 31, 17, 23)$

which is the product of 12 cycles each of order 4 and two of transpositions.

Let $\alpha_1 = ((XY)^6[X, Y]^5)^{18}$. Then

$\alpha_1 =$ (31, 62, 93, 124, 155, 186, 217,248,279,310,341,372,403),

which is a cycle of order 13. Let $\alpha_2 = \alpha_1^{-1}X$.

It is easy to show that

α_2= (1, 2, 3, . . . , 31)(32, 33, 34, . . . , 62) . . . (373,374, 375, . . . , 403),

which is the product of 13 cycles each of order 31.

Let: $\beta_1 = (Y^2)^{(XY)^{18}}$= (9, 20)(12, 23)(31, 53)(34, 56), $\beta_2 = \beta_1\ Y^{-1}$=(1, 9, 12, 20)(2, 6)(4, 5) (7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71) (73, 74), $\beta_3 = (Y^3\beta_2)^2$ =(1, 45)(12, 23), $\beta_4 = \beta_3^{(\alpha_2^{-1}\alpha_1^3)}$ =(31, 124)(155, 186) and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}}$ = (31, 372)(124, 155).

Let α_3=$\beta_5^{\beta_3^{(\alpha_2^{-1}\alpha_1)}}$. Hence

$\alpha 3$ =(31, 62)(93, 155).

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that

$\alpha_4 = (1,9)(2,6)(4,5)(7,8)(12,20)(13,17)(15,16)(18,19)(23,31)(24,28)$ $(26,27)(29,30)(34,42)(35,39)(37,38)(40,41)(45,53)(46,50)(48,49)(51,52)(56,64)$ $(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74)$,

which is a product of twenty eight transpositions.

Let $K = \prec \alpha_{2,}\alpha_{4\succ}$. Let $\theta : K \to PSL(3,5)$ be the mapping defined by

θ(31i+j) = j, $\forall 0 \le i \le 12$, $\forall\ 0 \le j \le 31$.

Since $\theta(\alpha_2)$ = (1, 2, ... , 31) and $\theta(\alpha_4)$ = (1, 9)(2, 6)(4, 5)(7, 8), then K$\approxeq$ $\theta(K) = PSL(3,5)$. Let $H_0 = \prec \alpha_1$, $\alpha_3 \succ$. Then $H_0 \approxeq PSL(3,3)$. Moreover, K conjugates H_0 into H_1 , H_1 into H_2 and so it conjugates H_{31} into H_0 , where

$H_i = \prec (i, 31+i, 62+i,\ 93+i, 124+i, 155+i, 186+i, 217+i, 248+i, 279+i, 310+i, 341+i, 372+i,)(i, 31+i)(62+i, 124+i) \succ\ \ \forall\ 0 \le i \le 30$

" . Hence we get $PSL(3,3)wr\ PSL(3,5) \subseteq G$. On the other hand, since

X =$\alpha_1\alpha_2$ and Y =$\alpha_4\alpha_3^X$ then $G \subseteq PSL(3,3)wr\ PSL(3,5)$.

Hence $G = PSL(3,3)wr\ PSL(3,5)$

.$\Diamond$

THEOREM 3.2 The wreath product $(PSL(3,3)wr\ PSL(3,5))wrC_K$ can be generated using two permutations, the first is of order $403k$ and an involution, for all integers $K \succeq 1$.

Proof :

Let $\sigma = (1, 2, ..., 403k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$ $(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$ $(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$ $(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$,

If k=1, then we get the group$PSL(3,3)wr\ PSL(3,5)$ which can be considered as the trivial wreath product $PSL(3,3)wr\ PSL(3,5)$ wr<id>. Assume that $k \succ 1$. Let $\alpha = \Pi_{i=0}^{31}\tau^{\sigma^{ik}}$, we get an element

$\delta = \alpha^{45}$ =(k, 2k, 3k, ... , 403k). Let $G_I = \prec\ \delta^{\sigma^i}\ , \tau^{\sigma^i} \succ$ be the groups acts on the sets Γ_i ={ i, k+i, 2k+i, ... , 402k+i }, for all 1$\le i \le k$. Since $\cap_{i=1}^{k}\Gamma_i = \phi$, then we get the direct product $G_1 \times G_2 \times ... \times G_k$,where, by Theorem 3.1 each$G_i \approxeq PSL(3,3)wr\ PSL(3,5)$. Let $\beta = \delta^{-1}\sigma$ =(1, 2, ... , k)(k+1, k+2, ... , 2k) ... (76k+1, 76k+2, ... , 403k). Let $H = \prec \beta \succ\approxeq$

C_k .H conjugates G_1 into G_2 ,G_2 into G_3 , . . . andG_k into G_1. Hence we get the wreath product $(PSL(3,3)wr\ PSL(3,3))wrC_K \subseteq G$. On the other hand, since $\delta\beta =$ (1, 2, . . . , k, k+1, k+2, . . . , 2k, . . . , 402k+1, 402k+2, . . . , 403k) $=\sigma$, then $\sigma \in (PSL(3,3)wr\ PSL(3,5))wrC_K$. Hence $G = \prec \sigma, \tau \succ \approxeq (PSL(3,3)wr\ PSL(3,5))wrC_K$.$\Diamond$

THEOREM 3.3 The wreath product $(PSL(3,3)wr\ PSL(3,5))wrS_K$ can be generated by using three permutations, the first is of order 403k, the second and the third are involutions , for all K$\geq$ 2.

Proof: Let $\sigma = (1, 2, ..., 403k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$
$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$
$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$
$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$, $\mu = (k, a)(2k, k{+}a)(3k, 2k+ a)...(403k + 402k + a)$.

since by theorem 3.2 $\prec \sigma, \tau \succ \approxeq (PSL(3,3)wr\ PSL(3,5))wrC_k$ and $(1, 2, ..., k)$ $(k + 1, ..., 2k)...(402 + 1, ..., 403k) \in$ $(PSL(3,3)wr\ PSL(3,5))wrC_K$ then $\prec$ $(1, ..., k)(k + 1, ..., 2k)...$
$(402k{+}1, ..., 403k, \mu \succ \approxeq S_k$.Hence $G = \prec \sigma, \tau, \mu \succ \approxeq (PSL(3,3)wr\ PSL(3,5))wrS_{k.}$
$\Diamond$

COROLLARY 3.4 The wreath product $(PSL(3,3)wrPSL(3,5))wrA_k$ can be generated by using three permutations, the first is of order 403k, the second is an involution and the third is of order 3, for all odd integers $k \geq 3$.

Proof : The proof is similar to the previous one. $\Diamond$

THEOREM 3.5 The wreath product $(PSL(3,3)wr\ PSL(3,5))wr(S_m wrC_a)$ can be generated by using three permutations, the first is of order 403k, the second and the third are involutions, where $k = am$ be any integer with $1 < a < k$.

Proof : Let $\sigma = (1, 2, ..., 403k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$
$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$
$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$
$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$, $\mu = (k, a)(2k, k{+}a)(3k, 2k+ a)...(403k + 402k + a)$.

since by theorem 3.2 $\prec \sigma, \tau \succ \approxeq (PSL(3,3)wr\ PSL(3,5))wrC_k$ and $(1, ..., k)$ $(k+1, ..., 2k)...(402+1, ..., 403k) \in (PSL(3,3)wr\ \ PSL(3,5))wrC_K$ then $\prec$ $(1, ..., k)(k+1, ..., 2k)...(402k+1, ..., 403k, \mu \succ \approxeq (S_m wr C_a)$.Hence $G = \prec$ $\sigma, \tau, \mu \succ \approxeq (PSL(3,3)wr\ PSL(3,5))wr(S_m wr C_a).\Diamond$

THEOREM 3.6 S_{403K+1} and A_{403K+1}can be generated using the wreath product $(PSL(3,3)wr\ PSL(3,5))wrC_k$ and a transposition in S_{403K+1} for all integers k >1 and an element of order 3 in A_{403K+1} for all odd integrs k > 1.

Proof : Let $\sigma = (1, 2, ..., 403k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$
$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$
$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$
$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$,

$\mu = (403k+1, 1)$ and $\mu^{\backslash} = (1, k, 403k+1)$ be four Permutations, of order 403k,2,2 and 3 respectively.

Let $H =< \sigma, \tau >$ .By theorem 3.2 $H \approxeq (PSL(3,3)wr\ PSL(3,5))wrC_K$.

Case 1: Let $G =< \sigma, \tau, \mu^{\backslash} >$.Let $\alpha = \sigma\mu$, then $\alpha = (1, 2, ..., 403k, 403k+1)$ which is a cycle of order 403k+1. By theorem 2.6 $G =< \sigma, \tau, \mu >=< \alpha, \mu > \approxeq S_{403K+1}$.

Case 2: Let $G =< \sigma, \tau, \mu^{\backslash} >$ .By theorem 2.7 $< \sigma, \mu^{\backslash} > \approxeq A_{403K+1}$.Since τ is an even Permutation, then $G \approxeq A_{403K+1}$.

THEOREM 3.7 .S_{403K+1} and A_{403K+1}can be generated using the wreath product $PSL(3,3)wr\ PSL(3,5)$ and an element of order k+1 inS_{403K+1} and A_{403K+1} for all integers k $\geq$ 1.

Proof : Let $G =< \sigma, \tau, \mu >$, Where

$$\begin{aligned}\sigma &= (1,2,3,...,403)(403(k-(k-1))+1,...,403(k-(k-1))+403)\\ &\quad ...(403(k-1)+1,...,403(k-1)+403),\end{aligned}$$

$$\begin{aligned}\tau &= (1,9)(2,6)(4,5(7,8)(12,20,23,31)(13,17)(15,16)(18,19)(24,28)\\ &\quad (26,27)(29,30)(34,42,56,64,)(35,39)(37,38)(40,41)(45,53)(46,50)\\ &\quad (48,49)(51,52)(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74)...\\ &\quad (403(k-1)+1,403(k-1)+9)...(403(k-1)+73,403(k-1)+74),\end{aligned}$$

$\mu = (403, 154, ..., 403k, 403k+1)$,Where $k - i. > 0$, be three permutations of

order $403, 4$ and $k+1$ respectively.
Let $H =< \sigma, \tau >$ .Define the mapping θ as follows

$$\theta(31(k-i)+j) = j \ \forall 1 \leq j \leq 31$$

$H =< \sigma, \tau > \approxeq PSL(3,3)wr\ PSL(3,5)$.Let $\alpha = \mu\sigma$ it is easy to show that $\alpha = (1, 2, ..., 403k, 403k+1)$, Which is acycle of order 403k+1.
Let $\mu\iota = \mu^{\sigma} = (1, 404, ..., 403(k-1)+1, 403k+1)$ and $\beta = [\mu, \mu^{\backslash}] = (1, 403, 403k+1)$. Since $h.c.f(1, 403, 403k+1) = 1$, then by theorem 2.5 $G =< \sigma, \tau, \mu > \approxeq S_{403K+1}$ or A_{403K+1} depending on whether k is an odd or an even integer respectively.$\diamondsuit$.

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