On a Type of Para-Kenmotsu Manifold

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Abstract. The object of this paper is to study a class of almost para-contact metric manifold namely para-Kenmotsu (briefly p-Kenmotsu) manifold in which $R(X, Y).C = 0$ where $C$ is the conformal curvature tensor of the manifold and $R$ is the Riemannian curvature and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors $X$ and $Y$.

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1. Introduction

Sato [1] defined the notions of an almost para-contact Riemannian manifold. After that, T. Adati and K. Matsumoto [2] defined and studied para-Sasakian and SP-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [3] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [4] have defined a class of almost para-contact metric manifolds namely para-Kenmotsu (briefly p-Kenmotsu) and SP-Kenmotsu manifolds. They also have studied the curvature properties of p-Kenmotsu manifold and the curvature properties of semi-symmetric metric connection of SP-Kenmotsu manifold.

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Let $M_n$ be an $n$-dimensional differentiable manifold equipped with structure tensors $(\phi, \xi, \eta)$ where $\phi$ is a tensor of type $(1,1)$, $\xi$ is a vector field, $\eta$ is a 1-form such that
\begin{align*}
\eta(\xi) &= 1 \\ \phi^2(X) &= X - \eta(X)\xi; \quad \bar{X} = \phi X
\end{align*}
Then $M_n$ is called an almost para-contact manifold.

Let $g$ be the Riemannian metric satisfying such that, for all vector fields $X$ and $Y$ on $M$,
\begin{align*}
g(X, \xi) &= \eta(X) \\ \phi \xi &= 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = n - 1 \\ g(\phi X, \phi Y) &= g(X,Y) - \eta(X) \eta(Y).
\end{align*}
Then the manifold $M_n$ is said to admit [1] an almost para-contact Riemannian structure $(\phi, \xi, \eta, g)$.

A manifold of dimension `$n$ with Riemannian metric `$g$' admitting a tensor field `$\phi$' of type $(1,1)$, a vector field `$\xi$' and a 1-form `$\eta$' satisfying (1.1), (1.3) along with
\begin{align*}
(\nabla_X \eta)Y - (\nabla_Y \eta)X &= 0 \\ (\nabla_X \nabla_Y \eta)Z &= [-g(X,Z) + \eta(X) \eta(Z)] \eta(Y) + [-g(X,Y) + \eta(X) \eta(Y)]\eta(Z) \\ \nabla_X \xi &= \phi^2 X = X - \eta(X)\xi
\end{align*}
is called a para-Kenmotsu manifold or briefly P-Kenmotsu manifold [4]. This paper deals with a type of p-Kenmotsu manifold in which
\begin{equation}
R(X, Y)C = 0
\end{equation}
where $C$ is the conformal curvature tensor of the manifold and $R$ is the Riemannian curvature. Let $(M_n, g)$ be an $n$-dimensional Riemannian manifold admitting a tensor field `$\phi$' of type $(1,1)$, a vector field `$\xi$' and a 1-form `$\eta$' satisfying
\begin{align*}
(\nabla_X \eta)Y &= g(X,Y) - \eta(X) \eta(Y) \\ g(X, \xi) &= \eta(X) \text{ and } (\nabla_X \eta)Y = \phi (\bar{X}, Y), \text{ where } \phi \text{ is an associate of } \phi
\end{align*}
is called a special P-Kenmotsu manifold or briefly SP-Kenmotsu manifold [4]. In this paper it is proved that if in a P-Kenmotsu manifold $(M_n, g)$ $(n > 3)$ the relation (1.9) holds then the manifold is conformally flat and hence is an SP-Kenmotsu manifold. Also it is shown that a conformally symmetric P-Kenmotsu manifold $(M_n, g)$ is an SP-Kenmotsu manifold for $n > 3$. (since it is known that $C = 0$ when $n = 3$, it has taken that $n > 3$).
It is known that [4] in a P-Kenmotsu manifold the following relations hold:

\[ S(X, \xi) = -(n-1) \eta(X) \]  
\[ g[R(X, Y)Z, \xi] = \eta[R(X, Y, Z)] = g(X, Z) \eta(Y) - g(Y, Z) \eta(X) \]  
\[ R(X, \xi) = -1 \]  
\[ R(X, \xi, X) = \xi \]  
\[ R(\xi, X, \xi) = X \]  
\[ R(X, Y, \xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ is orthogonal to } \xi. \]

where \( S \) is the Ricci tensor and \( R \) is the Riemannian curvature.

Moreover, it is also known that [4] a P-Kenmotsu manifold cannot be flat and a projectively flat P-Kenmotsu manifold i.e., a P-Kenmotsu manifold satisfying \( R(X, Y).W = 0 \) is said to be Einstein manifold with the constant curvature \(-n(n-1)\). The above results will be used in the next section.

### 2. P-Kenmotsu manifold satisfying \( R(X, Y).C = 0 \)

We have

\[ C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2} [g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y] + \]

\[ \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] \]  

where ‘r’ is the scalar curvature and ‘Q’ is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor ‘S’ [5] i.e.,

\[ g(QX, Y) = S(X, Y). \]

Then

\[ \eta(C(X, Y)Z) = g(C(X, Y)Z, \xi) \]

\[ = \frac{1}{n-2} \left[ \left( \frac{r}{n-1} + 1 \right) \left( g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \right) - \left( S(Y, Z)\eta(X) - S(X, Z)\eta(Y) \right) \right]. \]

Putting \( Z = \xi \) in (2. 3), we get

\[ \eta(C(X, Y) \xi) = 0 \]
Again putting $X = \xi$ in (2.3), we get
\begin{equation}
\eta (C(\xi, Y)Z) = \frac{1}{n-2} \left[ \left( \frac{r}{n-1} + 1 \right) g(Y, Z) - S(Y, Z) - \left( \frac{r}{n-1} + n \right) \eta(Y)\eta(Z) \right].
\end{equation}

Now
\begin{equation}
\end{equation}

In virtue of (1.9) we get
\begin{equation}
\end{equation}

Therefore
\begin{equation}
g(R(\xi, Y)C(U, V)W, \xi) - g(C(R(\xi, Y)U, V)W, \xi) - g(C(U, R(\xi, Y)V)W, \xi) - g(C(U, V)R(\xi, Y)W, \xi) = 0.
\end{equation}

From this, it follows that
\begin{equation}
\eta(Y)\eta(C(U, V)W) + \eta(U)\eta(C(Y, V)W) + \eta(V)\eta(C(U, Y)W) + g(Y, U)\eta(C(\xi, V)W) - g(Y, V)\eta(C(U, \xi)W) = 0.
\end{equation}

where
\begin{equation}
\eta(Y)\eta(C(U, V)W) + \eta(U)\eta(C(Y, V)W) + \eta(V)\eta(C(U, Y)W) + g(Y, U)\eta(C(\xi, V)W) - g(Y, V)\eta(C(U, \xi)W) = 0.
\end{equation}

Let \{ $e_i$ \}, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point. Then the sum $1 \leq i \leq n$ of the relation (2.9) for $U = e_i$ gives
\begin{equation}
\eta(C(\xi, V)W) = 0.
\end{equation}

By using (2.4), we have from (2.8)
\begin{equation}
\eta(Y)\eta(C(U, V)W) + \eta(U)\eta(C(Y, V)W) + \eta(V)\eta(C(U, Y)W) + g(Y, U)\eta(C(\xi, V)W) - g(Y, V)\eta(C(U, \xi)W) = 0.
\end{equation}

In virtue of (2.5) and (2.10) we have
\begin{equation}
S(V, W) = \left( \frac{r}{n-1} + 1 \right) g(V, W) - \left( \frac{r}{n-1} + n \right) \eta(V)\eta(W).
\end{equation}
Using (2.3), (2.4) and (2.12) the relation (2.11) reduces to
\[ C(U, V, W, Y) = 0. \] (2.13)
From (2.13) it follows that
\[ C(U, V) W = 0. \] (2.14)
Thus we can state the following theorem:

**Theorem 1:** A P-Kenmotsu manifold \((M_n, g)\) \((n > 3)\) satisfying the relation \(R(X, Y).C = 0\) is conformally flat and hence is an SP-Kenmotsu manifold. For a conformally symmetric Riemannian manifold, we have \(\nabla C = 0\) \([6]\) and hence for such a manifold \(R(X, Y).C = 0\) holds.

Thus we have the following corollary of the above theorem:

**Corollary 1:** A conformally symmetric P-Kenmotsu manifold \((M_n, g)\) \((n > 3)\) is an SP-Kenmotsu manifold.

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**References**

1. I. Sato, On a structure similar to the almost contact structure, Tensor (N.S.), 30 (1976), 219–224.

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