On a Type of Para-Kenmotsu Manifold

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Abstract. The object of this paper is to study a class of almost para-contact metric manifold namely para-Kenmotsu (briefly p-Kenmotsu) manifold in which $R(X, Y).C = 0$ where C is the conformal curvature tensor of the manifold and R is the Riemannian curvature and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X and Y .

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1. Introduction

Sato [1] defined the notions of an almost para-contact Riemannian manifold. After that, T. Adati and K. Matsumoto [2] defined and studied para-Sasakian and SP-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [3] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [4] have defined a class of almost para-contact metric manifolds namely para-Kenmotsu (briefly p-Kenmotsu) and SP-Kenmotsu manifolds. They also have studied the curvature properties of p-Kenmotsu manifold and the curvature properties of semi-symmetric metric connection of SP-Kenmotsu manifold.

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Let M_n be an n -dimensional differentiable manifold equipped with structure tensors (ϕ, ξ, η) where ϕ is a tensor of type $(1,1)$, ξ is a vector field, η is a 1-form such that

$$\eta(\xi) = 1 \quad (1.1)$$

$$\phi^2(X) = X - \eta(X)\xi; \quad \bar{X} = \phi X \quad (1.2)$$

Then M_n is called an almost para-contact manifold.

Let 'g' be the Riemannian metric satisfying such that, for all vector fields X and Y on M ,

$$g(X, \xi) = \eta(X) \quad (1.3)$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = n - 1 \quad (1.4)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (1.5)$$

Then the manifold M_n is said to admit [1] an almost para-contact Riemannian structure (ϕ, ξ, η, g) .

A manifold of dimension 'n' with Riemannian metric 'g' admitting a tensor field 'φ' of type $(1,1)$, a vector field 'ξ' and a 1-form 'η' satisfying (1.1), (1.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \quad (1.6)$$

$$(\nabla_X \nabla_Y \eta)Z = [-g(X, Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z) \quad (1.7)$$

$$\nabla_X \xi = \phi^2 X = X - \eta(X)\xi \quad (1.8)$$

is called a para-Kenmotsu manifold or briefly P-Kenmotsu manifold [4]. This paper deals with a type of p-Kenmotsu manifold in which

$$R(X, Y).C = 0 \quad (1.9)$$

where C is the conformal curvature tensor of the manifold and R is the Riemannian curvature.

Let (M_n, g) be an n -dimensional Riemannian manifold admitting a tensor field 'φ' of type $(1,1)$, a vector field 'ξ' and a 1-form 'η' satisfying

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) \quad (1.10)$$

$$g(X, \xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \phi(\bar{X}, Y), \text{ where } \phi \text{ is an associate of } \phi \quad (1.11)$$

is called a special P-Kenmotsu manifold or briefly SP-Kenmotsu manifold [4]. In this paper it is proved that if in a P-Kenmotsu manifold (M_n, g) ($n > 3$) the relation (1.9) holds then the manifold is conformally flat and hence is an SP-Kenmotsu manifold. Also it is shown that a conformally symmetric P-Kenmotsu manifold (M_n, g) is an SP-Kenmotsu manifold for $n > 3$. (since it is known that $C = 0$ when $n = 3$, it has taken that $n > 3$).

It is known that [4] in a P-Kenmotsu manifold the following relations hold:

$$S(X, \xi) = -(n-1)\eta(X) \quad (1.12)$$

$$g[R(X, Y)Z, \xi] = \eta[R(X, Y, Z)] = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \quad (1.13)$$

$$R(X, \xi) = -1 \quad (1.14)$$

$$R(X, \xi, \xi) = -X + \eta(X)\xi \quad (1.15)$$

$$R(X, \xi, X) = \xi \quad (1.16)$$

$$R(\xi, X, \xi) = X \quad (1.17)$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ is orthogonal to } \xi. \quad (1.18)$$

where S is the Ricci tensor and R is the Riemannian curvature.

Moreover, it is also known that [4] a P-Kenmotsu manifold cannot be flat and a projectively flat P-Kenmotsu manifold i.e., a P-Kenmotsu manifold satisfying $R(X, Y).W = 0$ is said to be Einstein manifold with the constant curvature $-n(n-1)$. The above results will be used in the next section.

2. P-Kenmotsu manifold satisfying $R(X, Y).C = 0$

We have

$$\begin{aligned} C(X, Y)Z = & R(X, Y)Z - \frac{1}{n-2} [g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y] + \\ & \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] \end{aligned} \quad (2.1)$$

where 'r' is the scalar curvature and 'Q' is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor 'S' [5] i.e.,

$$g(QX, Y) = S(X, Y). \quad (2.2)$$

Then

$$\begin{aligned} \eta(C(X, Y)Z) &= g(C(X, Y)Z, \xi) \\ &= \frac{1}{n-2} \left[\left(\frac{r}{n-1} + 1 \right) (g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) - (S(Y, Z)\eta(X) - S(X, Z)\eta(Y)) \right]. \end{aligned} \quad (2.3)$$

Putting $Z = \xi$ in (2.3), we get

$$\eta(C(X, Y)\xi) = 0 \quad (2.4)$$

Again putting $X = \xi$ in (2. 3), we get

$$\eta(C(\xi, Y)Z) = \frac{1}{n-2} \left[\left(\frac{r}{n-1} + 1 \right) g(Y, Z) - S(Y, Z) - \left(\frac{r}{n-1} + n \right) \eta(Y)\eta(Z) \right]. \quad (2. 5)$$

Now

$$\begin{aligned} (R(X, Y).C)(U, V)W &= R(X, Y)C(U, V)W - C(R(X, Y)U, V)W \\ &\quad - C(U, R(X, Y)V)W - C(U, V)R(X, Y)W. \end{aligned}$$

In virtue of (1.9) we get

$$\begin{aligned} R(X, Y)C(U, V)W - C(R(X, Y)U, V)W \\ - C(U, R(X, Y)V)W - C(U, V)R(X, Y)W = 0. \end{aligned} \quad (2. 6)$$

Therefore

$$\begin{aligned} g(R(\xi, Y)C(U, V)W, \xi) - g(C(R(\xi, Y)U, V)W, \xi) \\ - g(C(U, R(\xi, Y)V)W, \xi) - g(C(U, V)R(\xi, Y)W, \xi) = 0. \end{aligned} \quad (2. 7)$$

From this, it follows that

$$\begin{aligned} 'C(U, V, W, Y) - \eta(Y)\eta(C(U, V)W) + \eta(U)\eta(C(Y, V)W) + \eta(V)\eta(C(U, Y)W) \\ + \eta(W)\eta(C(U, V)Y) - g(Y, U)\eta(C(\xi, V)W) - g(Y, V)\eta(C(U, \xi)W) \\ - g(Y, W)\eta(C(U, V)\xi) = 0 \end{aligned} \quad (2. 8)$$

where

$$'C(U, V, W, Y) = g(C(U, V)W, Y).$$

Putting $Y = U$ in (2.8) we get

$$\begin{aligned} C(U, V, W, U) - \eta(U)\eta(C(U, V)W) + \eta(U)\eta(C(U, V)W) + \eta(V)\eta(C(U, U)W) \\ + \eta(W)\eta(C(U, V)U) - g(U, U)\eta(C(\xi, V)W) - g(U, V)\eta(C(U, \xi)W) \\ - g(U, W)\eta(C(U, V)\xi) = 0. \end{aligned} \quad (2. 9)$$

Let $\{e_i\}$, $i = 1, 2, \dots, n$ be an orthonormal basis of the tangent space at any point. Then the sum $1 \leq i \leq n$ of the relation (2. 9) for $U = e_i$ gives

$$\eta(C(\xi, V)W) = 0. \quad (2.10)$$

By using (2. 4), we have from (2. 8)

$$\begin{aligned} 'C(U, V, W, Y) - \eta(Y)\eta(C(U, V)W) + \eta(U)\eta(C(Y, V)W) + \eta(V)\eta(C(U, Y)W) \\ + \eta(W)\eta(C(U, V)Y) - g(Y, U)\eta(C(\xi, V)W) - g(Y, V)\eta(C(U, \xi)W) = 0 \end{aligned} \quad (2.11)$$

In virtue of (2. 5) and (2. 10) we have

$$S(V, W) = \left(\frac{r}{n-1} + 1 \right) g(V, W) - \left(\frac{r}{n-1} + n \right) \eta(V)\eta(W). \quad (2.12)$$

Using (2.3), (2.4) and (2.12) the relation (2.11) reduces to

$$'C(U, V, W, Y) = 0. \quad (2.13)$$

From (2.13) it follows that

$$C(U, V) W = 0. \quad (2.14)$$

Thus we can state the following theorem:

Theorem 1: A P-Kenmotsu manifold (M_n, g) ($n > 3$) satisfying the relation $R(X, Y).C = 0$ is conformally flat and hence is an SP-Kenmotsu manifold. For a conformally symmetric Riemannian manifold, we have $\nabla C = 0$ [6] and hence for such a manifold $R(X, Y).C = 0$ holds.

Thus we have the following corollary of the above theorem:

Corollary 1: A conformally symmetric P-Kenmotsu manifold (M_n, g) ($n > 3$) is an SP-Kenmotsu manifold.

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