Pseudo-Commutators in BCK-Algebras

Ardavan Najafi

Department of Mathematics
Islamic Azad University
Behbahan Branch, Behbahan, Iran
najafi2005@yahoo.com

Abstract

In this paper, we introduced the concept of pseudo-commutators in BCK-algebras and then we state and prove some related theorems on these notions.

Mathematics Subject Classification: Primary: 06F35 ; Secondary: 08A05, 03G25.

Keywords: BCK-Algebra, pseudo-commutators, derived subalgebra

1 Introduction

By an algebra $G = (G, ., 0)$ we main a non-empty set $G$ together with a binary multiplication and a some distinguished element $0$. In 1966, Y. Imai and K. Iseki [2] defined a class of algebras of type $(2,0)$ called BCK-algebra which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation. on the other hand the notion of implication algebra [1]. We can define an implication in each BCK-algebra by $y \rightarrow x = xy$. So, we can see (.) as the dual of implication of B-C-K-logic. in this paper, a binary multiplication will be denoted by juxtaposition. we use dots only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy.zy)xz = 0$.

Definition 1.1. An algebra $(G, ., 0)$ is called a BCK-algebra. If for all $x, y, z \in G$, the following axiom hold :
\[(BCK1) \quad (xy.xz).zy = 0, \]
\[(BCK2) \quad (x.(xy)).y = 0, \]
\[(BCK3) \quad xx = 0, \]
\[(BCK4) \quad 0x = 0, \]
\[(BCK5) \quad xy = yx = 0 \text{ implies } x = y. \]

The above definition is a dual form of the ordinary definition\[1, 3\]. on any BCK-algebra \((G, ., 0)\) for all \(x, y \in G\), we can define the natural order putting
\[x \leq y \iff xy = 0 \quad (1)\]
It is not difficult to verify that this order is partial and 0 is its smallest element. In a BCK-algebra \((G, ., 0)\) for all \(x, y, z \in G\) the following identities hold.
\[(1.2) \quad x0 = x \quad \text{and} \quad x \leq y \text{ implies } xz \leq yz \text{ and } zy \leq zx \text{ and } xy.z = xz.y\]
A BCK-algebra \(G\) is said to be bounded if there exists an elements 1 \(\in G\) such that \(x \leq 1\) for all \(x \in G\). For elements \(x\) and \(y\) of a BCK-algebra \(G\), we denote
\[(1.3) \quad x \wedge y = y, (yx) \quad \text{and} \quad x \vee y = N(Nx \wedge Ny) \text{ where } Nx = 1x.\]
A BCK-algebra \(G\) is said to be commutative if it satisfies \(x \wedge y = y \wedge x\) for all \(x, y \in G\). A non-empty subset \(S\) of a BCK-algebra \(G\) is called a BCK-subalgebra of \(G\), if \(xy \in S\) whenever \(x, y \in S\). Also a non-empty subset \(I\) of a BCC-algebra \(G\) is called and BCC-ideal if (i) 0 \(\in I\), (ii) \((xy).z \in I\) and \(y \in I\) imply \(xz \in I\) for all \(x, y \in G\). If (ii) holds only in case when \(z = 0\), then \(I\) is called a BCK-ideals. A BCK- algebra \(G\) is called implicative iff \(x.(yx) = x\), also \(G\) is called positive implicative BCK-algebra if it satisfy in property \(xz.yz = xy.z\), In any commutative BCK-algebra, the following statements hold:
\[(1.4) \quad x \wedge x = x \vee x = x. \]
\[(1.5) \quad x \vee 0 = 0 \vee x = x \wedge 1 = 1 \wedge x = x. \]
\[(1.6) \quad x \wedge y = y \wedge x. \]
\[(1.7) \quad x \vee y = y \vee x. \]
\[(1.8) \quad x \vee 1 = 1 \vee x = 1. \]
\[(1.9) \quad 0 \wedge x = x \wedge 0 = 0. \]
\[(1.10) \quad NNx=x. \]

2 Main Results

Definition 2.1. Let \((G, ., 0)\) be a BCK-algebra and let \(x, y, x_1, x_2, x_3, \ldots\)be elements of \(G\). then the element \((x_1 \wedge x_2).((x_2 \wedge x_1))\) of \(G\) is called pseudo-commutator of \(x_1\) and \(x_2\) and denoted by \([x_1, x_2]\). i.e.
\[[x_1, x_2] = (x_1 \wedge x_2).((x_2 \wedge x_1) \quad (2)\]

Lemma 2.2. Let \((G, ., 0)\) be a BCK-algebra. then for any \(x, y \in G\)
\(i)\) if \(x \leq y\) then \([x, y] = 0.\)
\(ii) [x, 0] = [0, x] = [x, x] = 0.\)
Proof. i) Let \( x \leq y \) then \( xy = 0 \), but \( [x, y] = (x \land y) \). (\( y \land x \)) \( = (y \cdot (yx)) \cdot (x \cdot (xy)) = (y \cdot (yx)) \cdot (x \cdot (xy)) = x = 0 \).

ii) \([x, 0] = (x \land 0) \cdot (0 \land x) \) \( = 0 \) and \([0, x] = (0 \land x) \cdot (x \land 0) \) \( = 0 \). But \([x, x] = (x \land x) \cdot (x \land x) \) \( = xx = 0 \).

Example 2.3. Let \( G = \{0, 1, 2, 3, 4\} \) and let the multiplication be defined by the following table

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((G, ., 0)\) is a bounded BCK-algebra with unit 4. The pseudo-commutator of 2 and 3 = [2, 3] = 0 because \( 2 \leq 3 \) and the pseudo-commutator of 3 and 2 = [3, 2] = (3 \( \land 2 \)) = (2.32) = (2.33) = (2.0)(3.3) = 20 = 2. Therefore [2, 3] \( \neq [3, 2] \). So in general case \([x, y] \neq [y, x] \).

It is useful to be able to form pseudo-commutators of subsets as well as elements.

Definition 2.4. Let \( X_1, X_2, \ldots, X_n \) be nonempty subsets of a BCK-algebra \( G \). Define the pseudo-commutator subalgebra of \( X_1 \) and \( X_2 \) to be

\[
[X_1, X_2] = \{ [x_1, x_2] | x_1 \in X_1, x_2 \in X_2 \}
\]

Also the subset \([G, G] = \{ [a, b] | a, b \in G \}\) of \( G \) is called derived subalgebra of \( G \) and we will denote \([G, G]\) by \( G' \).

\[
G' = [G, G] = \{ [a, b] | a, b \in G \}
\]

since for any \( x \in G \), we have \([x, 0] = [0, x] = [x, x] = 0 \). Therefore, \([A, B] \neq \phi \) for any two subalgebra \( A, B \) of \( G \), so \( 0 \in [A, B] \). That is always \( 0 \in G' \).

The following example shows that \( G' \) is not an ideal.

Example 2.5. Let \( G = \{0, a, b, c, d \} \) and \((.)\) operation be given by the following table

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>d</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>
then $(G,.,0)$ is a BCK-algebra. Consider that $A = \{0, a\}$ and $B = \{0, c\}$ are two subalgebra of $G$. It is easy to check that $[A,B] = \{0\}$ and $[B,A] = \{0,a\}$ therefore $[A,B] \neq [B,A]$. Now we see that $G' = \{0,a,b\}$ is a subalgebra of $G$, but $G'$ is not an ideal of $G$, because $db = b \in G'$ and $b \in G'$ but $d \notin G'$.

**Theorem 2.6.** If $(G,.,0)$ be a BCK-algebra. Then $G$ is commutative $\iff G' = \{0\}$.

**Proof.** Let $G$ be a commutative BCK-algebra. Let $x, y \in G$. Then $x \land y = y \land x$. Commutator of $x$ and $y = [x,y] = (x \land y).(y \land x) = (x \land y).(x \land y = 0$. Thus, $G'$ is generated by $0$. Therefore, $G' = \{0\}$. Conversely, Let $G' = \{0\}$. Then for a commutator of $x$ and $y = [x,y] = 0$ $[y,x] = 0$. But $[x,y] = (x \land y).(y \land x) = 0$ and $[y,x] = (y \land x).(x \land y) = 0$ so $x \land y = y \land x$. (by use $xy = 0 = yx$ imply $x = y$)

**Lemma 2.7.** If $(G,.,0)$ is a BCK-algebra. Then $G'$ is a sub-algebra of $G$.

**ACKNOWLEDGEMENTS.** The author wish to thank of family. This paper is the result of a research project. Cooperation with the research unit of Islamic Azad University Behbahan Branch.

**References**


Received: September, 2012.