Some Properties on Morphic Groups

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Abstract

The paper must have abstract. In this paper we continue the investigations on morphic groups. We also show that if a group is normally uniserial and of order $p^3$ with $p$ prime it must be morphic and so give a negative answer to one of the questions of [4]. We characterize the morphic groups of order $p^3$ with $p$ an odd prime. We also explore the set of subgroups of a morphic group which still morphic by restriction. We also show that if a group is normally uniserial and of order $p^3$ with $p$ prime it must be morphic.

Mathematics Subject Classification: 20E34

Keywords: morphic groups, morphic morphism, semidirect product, stability, cyclic

1 Introduction

A morphism $\alpha$ from a group $G_1$ to a group $G_2$ is called dualizable if there exists a morphism $\beta$ from $G_2$ to $G_1$ such $\ker \alpha = \beta(G_2)$ and $\ker \beta = \alpha(G_1)$. In cases where $G_1 = G_2$ these morphisms are called morphic. A group $G$ is called morphic if every endomorphism $\phi$ of $G$ for which $\phi(G)$ is normal in $G$ satisfies $G/\phi(G) \cong \ker \phi$. This condition for modules was introduced in 1976 by Gertrude Ehrlich [2]. It arose in her characterization of when the endomorphism ring of a module is unit regular. A group-theoretic version of
Ehrlich’s theorem is given in [3]. The condition $M/\phi(M) \simeq \ker\phi$ was studied in the context of rings in [5], and then for modules in [6]. In the present paper we continue the investigations of [4] on morphic groups. We give several examples of morphic and non morphic groups, generalize some results on finite groups and give an explicit proof showing that a dihedral group $D_n$ is morphic if and only if $n$ is odd. We also give a negative answer to the question whether an uniserial group of length three is morphic?

2 Preliminary Notes

If $G, G_1, G_2$ are groups, we write $\text{end}(G)$ (resp $\text{mor}(G_1, G_2)$) for the monoid of endomorphisms $\phi : G \to G$ (resp for the set of morphisms $\phi : G_1 \to G_2$); and we write $\text{aut}(G)$ (resp $\text{iso}(G_1, G_2)$) for the group of automorphisms of $G$ (resp the set of isomorphisms from $G_1$ to $G_2$). As usual, we write $H \triangleleft G$ to indicate that $H$ is a normal subgroup of $G$; we write $Z = Z(G)$ for the centre of $G$; and we write $G'$ for the commutator (or derived) subgroup of $G$. We write $Z_n$ for the commutative groupe $\mathbb{Z}/n\mathbb{Z}, C_n$ for the cyclic group of order $n$; and $D_n$ for the dihedral group of order $2n$. If $H$ and $K$ are subgroups of $G$; we write $G = H \odot K$ to mean that $H \triangleleft G; K \triangleleft G; G = HK$ and $H \cap K = 1$ and in this case we say that $H$ and $K$ are direct factors of the group $G$. We say that $G$ is a semidirect product of $K$ by $H$; and we write $G = K \rtimes H$; if $K \triangleleft G; G = KH$ and $H \cap K = 1$ in this case we say that $K$ is a semidirect factor of $G$:

3 Morphic endomorphisms and morphic groups

We begin with some characterizations of the group morphisms and endomorphisms of interest here.

Definition 3.1 If $G_1, G_2$ are groups; $\phi \in \text{mor}(G_1, G_2)$ is called normal if $\phi(G_1)$ is normal in $G_2$.

Examples 3.2

1. If $H$ is a normal subgroup of a group $G$ then the inclusion map $i : H \to G$ is a normal morphism.

2. Every onto groups morphism is normal.

3. If in the definition $G_2$ is abelian then $\phi$ is normal.

Lemma 3.3.

If $G_1, G_2$ are groups and $\phi \in \text{mor}(G_1, G_2)$, the following are equivalent:

1. $\phi(G_1) \triangleleft G_2$ and $G_2/\phi(G_1) \simeq \ker\phi$. 


2. There exists \( \beta \in \text{mor}(G_2, G_1) \) with \( \ker \phi = \beta(G_2) \) and \( \phi(G_1) = \ker \beta \).

3. There exists \( \beta \in \text{mor}(G_2, G_1) \) with \( \ker \phi \simeq \beta(G_2) \) and \( \phi(G_1) = \ker \beta \).

**Proof 3.4.**

1. \( 1 \) \( \implies \) \( 2 \) Let \( f : G_2/\phi(G_1) \to \ker \phi \) be an isomorphism and set \( \beta : G_2 \to G_1 \) such \( \beta(g) = f(\overline{g}) \). It is easy to prove that \( \beta \) is well defined and a group morphism. We also have \( \beta(G_2) = f(G_2/\phi(G_1)) = \ker \phi \) because \( f \) is an isomorphism. In the other side \( \ker \beta = \{ g \in G_2 \mid \beta(g) = 1 \} = \{ g \in G_2 \mid f(\overline{g}) = 1 \} = \phi(G_1) \).

2. \( 2 \) \( \implies \) \( 3 \) is trivial.

3. \( 3 \) \( \implies \) \( 1 \) Let \( \beta : G_2 \to G_1 \) be a morphism as in \( 3 \), then \( G_2/\ker \beta \simeq \beta(G_2) \) but \( \phi(G_1) = \ker \beta \triangleleft G_2 \) and \( G_2/\phi(G_1) = G_2/\ker \beta \simeq \beta(G_2) \simeq \ker \phi \).

**Definition 3.5** \( \phi \in \text{end}(G) \) is called morphic if it satisfies one of the conditions of the lemma 2.2. We also have that every automorphism of \( G \) and the trivial endomorphism are morphic by (1) of the same lemma.

**Definition 3.6** A group \( G \) is called morphic if every endomorphism \( \phi \) of \( G \) for which \( \phi(G) \) is normal in \( G \) is morphic.

**Proposition 3.7** Let \( G_1, G_2 \) be two isomorphic groups. If \( G_1 \) is morphic so is \( G_2 \).

The proof bellow can seem redundant but we give it as an illustration.

**Proof 3.8** Let \( f : G_1 \to G_2 \) be an isomorphism and \( \beta : G_2 \to G_2 \) be an endomorphism such \( \beta(G_2) \triangleleft G_2 \). We have to prove that \( G_2/\beta(G_2) \simeq \ker \beta \), \( f^{-1} \circ \beta \circ f \) is in \( \text{end}(G_1) \) and is such \( f^{-1} \circ \beta \circ f(G_1) = f^{-1}(\beta(G_2)) \triangleleft G_1 \). But \( G_1 \) is morphic so

\[
G_1/(f^{-1} \circ \beta \circ f)(G_1) \simeq \ker(f^{-1} \circ \beta \circ f).
\]

Let

\[
\Psi : G_1 \to G_2/\beta(G_2), \quad x \mapsto \overline{f(x)}.
\]

\( \Psi \) is a group morphism and surjective so \( G_1/\ker \Psi \simeq G_2/\beta(G_2) \). But \( \ker \Psi = \beta(G_1) \) so, \( G_1/\beta(G_1) \simeq G_2/\beta(G_2) \).

Let \( x \in \ker(f^{-1} \circ \beta \circ f) \) so \( 1 = (f^{-1} \circ \beta \circ f)(x) = f^{-1}(\beta(f(x))) \) and \( \beta(f(x)) \in \ker f^{-1} = 1 \) and then \( f(x) \in \ker \beta \) so the group morphism :

\[
\Phi : \ker(f^{-1} \circ \beta \circ f) \to \ker \beta, \quad x \mapsto \overline{f(x)}.
\]
is well defined. In addition we have ker Φ = 1 and if y ∈ ker β and because y is
in G₂ there exists x ∈ G₁ such y = f(x). But (f⁻¹ ∘ β ∘ f)(x) = f⁻¹(β(f(x))) =
f⁻¹(β(y)) = f⁻¹(1) = 1 so x ∈ ker(f⁻¹ ∘ β ∘ f) and Φ is an isomorphism. In
conclusion we have :

\[ G₂/β(G₂) \cong G₁/f⁻¹(β(f(G₁))) \cong ker(f⁻¹ ∘ β ∘ f) \cong ker β. \]

And G₂ is then morphic.

The example below has been treated in [4] for the case n = 2

**Example 3.9** Let G denote the group \( C_n \times C_{2n} \), where \( C_n = \langle a \rangle \) and \( C_{2n} = \langle b \rangle \) then :

1. there exists a morphic endomorphism of G.
2. G is not morphic.
3. In Lemma 2.2, we cannot replace (3) by: “There exists β ∈ end(G) such that ker φ = β(G) and φ(G) ≃ ker β
4. The composite of morphic endomorphisms need not be morphic.

**Proof 3.10** 1. Let \( α, f : C_n \times C_{2n} \longrightarrow C_n \times C_{2n} \) such \( α(x, y) = (x, 1) \)
and \( f(x, y) = (1, y) \). It is easy to see that \( α, f ∈ end(G), ker α = 1 \times \)
\( C_{2n}, ker f = α(G) \) and so \( G/α(G) = G/ker f \cong f(G) = 1 \times C_{2n} = \)
ker α. α is then a morphic endomorphism.

2. Let \( f : C_n \longrightarrow C_{2n}; \ x = a^k \longrightarrow y = b^{2k} \) and \( α : C_n \times C_{2n} \longrightarrow \)
\( C_n \times C_{2n}; \ (a^k, b^l) \longmapsto (1, b^{2k}). \ b^{2k} = 1 \iff n|k \iff a^k = 1 \) so
ker α = 1 \times C_{2n} \cong C_{2n}. It is easy to see that \( Ψ : C_n \times 1 \longrightarrow 1 \times f(C_n) = \)
\( G/α(G); (a^k, 1) \longmapsto (1, b^{2k}) \) is an isomorphism. We also have \( G/α(G) = \)
\( \{(a^k, 1), (a^k, b) / 1 \leq k \leq n\} \cong C_n \times C_2 \) which is not isomorph to
\( C_{2n} = ker α \) because the second group is cyclic but the first is not.

3. let \( α : C_n \times C_{2n} \longrightarrow C_n \times C_{2n}; \ (a^k, b^l) \longmapsto (1, b^{2k}). \) and \( β : C_n \times C_{2n} \longrightarrow \)
\( C_n \times C_{2n}; \ (x, y) \longmapsto (1, y). \) We then have
\( C_n \times C_{2n}/α(G) \cong C_1 \times C_1 \) and ker α = \( C_1 \times C_{2n} \) and the groups are not
isomorphic and α is not morphic.

We then have two consequences.

**Lemma 3.11** [4]

1. If φ ∈ end(G) is morphic then φ is one to one if and only if it is onto.
Some properties on morphic groups

2. If $\phi \in \text{end}(G)$ is morphic, so also are $\phi \circ \psi$ and $\psi \circ \phi$ for every automorphism $\psi$ of $G$.

**Proof 3.12** 1. Since $G/\phi(G) \simeq \text{ker}\phi$; then $\text{ker}\phi = 1$ if and only if $\phi(G) = G$.

2. If $\psi \in \text{aut}(G)$ and $\phi \in \text{end}(G)$ is morphic, then $G/\psi \circ \phi(G) \simeq G/\phi(G) \simeq \ker\phi = \ker\psi \circ \phi$ and $G/\phi \circ \psi(G) = G/\phi(G) \simeq \ker\phi \simeq \psi^{-1}(\ker\phi) = \ker(\phi \circ \psi)$.

**Corollary 3.13** For all $n \in \mathbb{N}$; the only morphic endomorphisms of $n\mathbb{Z}$ are the trivial endomorphism.

**Proof 3.14** Let $f \in \text{end}(n\mathbb{Z}) \setminus \{x \mapsto 0, x \mapsto \pm x\}$, then there exists $a \in \mathbb{Z} \setminus \{0, \pm 1\}$ such $f(n) = an$. $f$ is then one to one and if it is morphic it must be onto. But this is false.

4 Subgroups and morphic property

**Proposition 4.1** Let $G$ be a group, $H$ be a subgroup of $G$ and $\alpha$ be a morphic endomorphism of $G$ such

1. $\alpha(H) \lhd H$,

2. $\forall \overline{x} \in G/\alpha(G); \exists x \in H \ / \ \overline{x} = \overline{xt}$,

3. $\forall x \in G \setminus H; \alpha(x) \notin H$;

then $\alpha_H$ the restriction of $\alpha$ to $H$ is also morphic.

**Proof 4.2** $\alpha$ is morphic then if $\alpha(G) \lhd G$ we have $G/\alpha(G) \simeq \text{ker}\alpha$. Let $f : G/\alpha(G) \longrightarrow \text{ker}\alpha$ be an isomorphism and

$$g : H/\alpha(H) \longrightarrow \text{ker}\alpha_H : \ h \longmapsto f(\overline{h})$$

where $\overline{h}$ is the class of $h$ in $G/\alpha(G)$. The map is well defined because $f(\overline{h})$ is in $\text{ker}\alpha_H$ elsewhere if $f(\overline{h}) \notin H \implies f(\overline{h}) \in G \setminus H$ and by the third condition $\alpha(f(\overline{h})) \notin H$ and then $\alpha(f(\overline{h})) \neq 1$ and this means that $f(\overline{h}) \notin \text{Ker}\alpha$ which is a contradiction.

Now if $h = k$ then $hk^{-1} \in \alpha(H) \subset \alpha(G)$ and $\overline{h} = \overline{k} \implies f(\overline{h}) = f(\overline{k})$.

Let $y \in \text{ker}\alpha_H \subset \text{ker}\alpha$ then there exists $x \in G / f(\overline{x}) = y$ but the second condition implies that there exists $x_1 \in H$ such $\overline{x} = \overline{x_1}$ so $g(x_1) = f(\overline{x_1}) = f(\overline{x}) = y$ and $g$ is onto.

Let $x_1, x_2 \in H/\alpha(H)$ such $g(x_1) = g(x_2)$ then $f(\overline{x_1}) = f(\overline{x_2}) \iff \overline{x_1} = \overline{x_2} \iff x_1.x_2^{-1} \in \alpha(G)$. But $x_1, x_2$ are in $H$ so $x_1.x_2^{-1} \in H$ and there exists
\(y \in G\) such \(x_1 x_2^{-1} = \alpha(y)\). If \(y \notin H\) then \(\alpha(y) \notin H\) and this contradicts the fact that \(x_1 x_2^{-1} \in H\) and so \(x_1 x_2^{-1} = \alpha(y) \in \alpha(H)\) and therefore \(\hat{x}_1 = \hat{x}_2\) and \(g\) is one to one.

Let \(\hat{x}_1, \hat{x}_2 \in H/\alpha(H)\); then

\[
g(\hat{x}_1 \cdot \hat{x}_2) = g(\hat{x}_1 \cdot \hat{x}_2) = f(\hat{x}_1 \cdot \hat{x}_2) = f(\tilde{x}_1) f(\tilde{x}_2) = g(\tilde{x}_1) g(\tilde{x}_2)
\]

### Proposition 4.3
Let \(G\) be a finite group, \(H\) be a subgroup of \(G\) and \(\alpha\) be a morphic endomorphism of \(G\) such

1. \(\alpha(H) \triangleleft H\),
2. \(|H/\alpha(H)| = |G/\alpha(G)|\),
3. \(H \cap \alpha(G) \subseteq \alpha(H)\);

then \(\alpha_H\) the restriction of \(\alpha\) to \(H\) is also morphic.

#### Proof 4.4
From the second condition, the fact that \(\alpha\) is morphic on \(G\) and the first isomorphism theorem we deduce that \(\ker \alpha_H = \ker \alpha\). Let

\[
\Phi : H/\alpha_H(H) \to G/\alpha(G)
\]

\[
\bar{x} \mapsto \bar{\tilde{x}}
\]

It is no hard to see that \(\Phi\) is well defined, a group morphism and one-to-one. So it is an isomorphism. In the other side \(G/\alpha(G) \cong \ker \alpha = \ker \alpha_H\) then \(H/\alpha_H(H) \cong \ker \alpha_H\) and \(\alpha_H\) is morphic.

#### Example 4.5
Conditions in proposition 3.2 are not necessary like proved below.

Take \(G = D_4, H = \langle a \rangle\) and \(\alpha \in \text{end}(G)\) defined by : \(\alpha(a) = a^2\) and \(\alpha(b) = 1\). We can prove that \(\alpha, \alpha_H\) are morphic but the second condition \(|G/\alpha(G)| = 4 \neq |H/\alpha_H(H)| = 2\) in proposition isn’t verified.

#### Corollary 4.6
If in the proposition 3.1; \(\alpha_H\) is onto; then \(\alpha\) is also onto.

#### Proof 4.7
\(\alpha_H\) is onto implies that \(H = \alpha(H) \subseteq \alpha(G)\). The condition 2) in the proposition implies that \(\forall x \in G; \ \exists x_1 \in H\) such \(\bar{x} = \bar{x}_1\). But \(\bar{x}_1 = \bar{T}\) implies that \(\bar{x} = \bar{T}\) and then \(x \in \alpha(G)\) and then there exists \(x' \in G\) such \(x = \alpha(x')\) and \(\alpha\) is onto.

### Proposition 4.8
Let \(p\) be a prime integer and \(G\) an uniserial group of order \(p^3\). If \(\alpha \in \text{end}(G)\) is nontrivial morphic endomorphism then \(\alpha^2 = 1\) elsewhere \(\ker \alpha^2 = \alpha(G)\).
Proof 4.9 Let $\alpha \in \text{end}(G)$ such $\alpha(G) \triangleleft G$. $\alpha$ is non-trivial then $|\alpha(G)| \in \{p, p^2\}$. By the first isomorphism theorem and the fact that $G$ is uniserial we must have $1 \triangleleft \ker \alpha \triangleleft \alpha(G) \triangleleft G$ or $1 \triangleleft \alpha(G) \triangleleft \ker \alpha \triangleleft G$.

1. Case: $1 \triangleleft \ker \alpha \triangleleft \alpha(G) \triangleleft G$. Let 

$$\Psi : G \longrightarrow \alpha(G)/\ker \alpha$$

$$x \longmapsto \alpha(x).$$

It is easy to prove that $\Psi$ is well defined and is onto and its kernel is exactly the set $\ker \alpha^2$. Applying the first isomorphism theorem we get $G/\ker \alpha^2 \cong \alpha(G)/\ker \alpha$ and then by Lagrange theorem the order of the subgroup $\ker \alpha^2$ is $p^2$ and also normal in $G$ which is uniserial so we get $\ker \alpha^2 = \alpha(G)$.

2. If $1 \triangleleft \alpha(G) \triangleleft \ker \alpha \triangleleft G$. One can remark that $\forall x \in G; \alpha^2(x) = 1$ and then $\alpha^2 = 1$.

Proposition 4.10 Let $p$ be a prime integer and $G$ an uniserial group of order $p^3$. If $\alpha \in \text{end}(G)$ is nontrivial such $\alpha(G)$ is of order $p^2$. Then $\alpha$ is morphic if and only if $\ker \alpha \triangleleft G$.

Proof 4.11 Assume that the order of $\alpha(G)$ is $p^2$ and normal in $G$ which is uniserial then we are in the case; $1 \triangleleft \ker \alpha \triangleleft \alpha(G) \triangleleft G$. Then $|G/\alpha(G)| = p = |\ker \alpha|$ so $G/\alpha(G)$ and $\ker \alpha$ are both of prime order then are cyclic and isomorphic to $\mathbb{Z}_p$ so they are isomorphic.

The converse is the definition of morphic endomorphisms.

Definition 4.12 Let $D_n = \langle a, b \rangle \; a^n = b^2 = 1, ba^k b = a^{-k} \rangle$ be the dihedral group and $Q_n = \langle a, b \rangle \; a^n = b^2 = (ab)^2 = 1 \rangle$ be the quaternion group each of order $2n$.

Proposition 4.13 [4] the quaternion group $Q_2$ is not morphic.

Proof 4.14 First we know that $Q_4$ admits the presentation: $Q_4 = \langle a, b \rangle \; a^4 = 1, a^2 = b^2, ab = ba^3 \rangle = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$; so if $\alpha : Q_4 \longrightarrow Q_4$; is an endomorphism of $Q_4$ such $\alpha(a) = 1, \alpha(b) = a^2$ then $\ker \alpha = \{1, a, a^2, a^3\}$ and $\alpha(Q_4) = \{1, a^2\}$. $\alpha(Q_4)$ is normal in $Q_4$ but is easy to see that $Q_4/\alpha(Q_4) = \{1, a^2, \overline{b}, \overline{ab}\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\ker \alpha \simeq \mathbb{Z}_4$.

Lemma 4.15 1. If $n$ is odd then the normal subgroups of $D_n$ are $D_n$ and the subgroups of $\langle a \rangle$.

2. If $n \ (n \neq 2)$ is even then the normal subgroups of $D_n$ are $D_n$, the subgroups of $\langle a \rangle$, the subgroup generated by $a^2$ and $b$ and the subgroup generated by $a^2$ and $ba$. 
The proposition bellow is proposition 15 in [4] with different proof.

**Proposition 4.16** The dihedral group $D_n$ is morphic if only if $n$ is odd.

**Proof 4.17** Let us take a nontrivial endomorphism $\alpha \in \text{end}(D_n)$ such $\alpha(D_n) < D_n$. By the lemma 3.9 , the only possible cases are $\alpha(D_n) \leq <a>$ or $\alpha(D_n) = D_n$.

1. If $\alpha(D_n) = <a>$ then $|\ker \alpha| = 2$ but $\ker \alpha$ must be a subgroup of $<a>$ so 2 divides $n$ and this contradicts the fact that $n$ is odd.

2. If $\alpha(D_n)$ and $\ker \alpha$ are a nontrivial subgroups of $<a>$ then if $|\alpha(D_n)| = m$ and $|\ker \alpha| = p$ we then have $n = mn_1$ and $n = pn_2$ and by the first isomorphism theorem we also have $2n = mp$ and those relations give $n = 2n_1n_2$ so $n$ is even which is impossible.

So $\alpha$ can’t be morphic and then the only morphic endomorphisms are the trivial ones.

Conversly suppose that $n$ is even so the endomorphism of $D_n$ given by:

$$
\alpha : D_n \rightarrow D_n \\
a \mapsto 1 \\
b \mapsto a^n
$$

is such $\ker \alpha = <a>$ and $\alpha(D_n) = <a^\frac{n}{2} > \leq <a>$ so by the lemma 3.9 $\alpha(D_n) < D_n$. But in this case

$$
D_n/\alpha(D_n) = \{1, a, \cdots, a^{\frac{n}{2}-1}, b, ab, \cdots, a^{\frac{n}{2}-1}b\}
$$

where the elements $b$ and $ab$ are of order two. In the other side $\ker \alpha = <a>$ contains only one element of order two which is $a^\frac{n}{2}$. So the two groups can’t be isomorphic.

**Remarks 4.18**

1. In the lemma if we take $n = 4$ then we have an example of uniserial group of length 3 which isn’t morphic

$$
(1 < <a^2 > <a> <a^3>)
$$

and so give a negative answer to the question of Yuanlin.Li, W.K.Nikolson and Libo Zan [4]

2. In the case where $n$ is even like proved in the above example this is’nt also true.
Example 4.19 Let $D_4 = \langle a, b/a^4 = b^2 = 1, bab = a^3 \rangle$ and $\alpha_1, \alpha_2, \alpha_3 \in \text{end}(D_4)$ such $\alpha_1(a) = 1, \alpha_2(b) = a^2, \alpha_2(a) = a^2, \alpha_2(b) = a^2$. It is easy to see that in all cases $\alpha_i(D_4) = \{1, a^2\} \triangleleft D_4$ and $D_4/\alpha_i(D_4) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. In the other side $\text{ker}\alpha_i(D_4) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ only for $i = 2, 3$; $\text{ker}\alpha_1(D_4) \simeq \mathbb{Z}_4$. So $D_4$ isn’t morphic.

Definition 4.20 A subgroup $H$ of a group $G$ is called strongly characteristic if its image under every endomorphism of the group $G$ is normal in $H$.

Proposition 4.21 Let $H$ be a strongly characteristic subgroup of the group $G$. If $\alpha$ is a morphic endomorphism of $G$ such $\text{ker}\alpha \subset H$ and $H \cap \alpha(G) \subset \alpha(H)$; then its restriction to $H$ is also morphic.

Proof 4.22 Let $f : G/\alpha(G) \to \text{ker}\alpha$

\[
\overline{x} \mapsto f(\overline{x})
\]

be an isomorphism associated to $\alpha$ and let:

\[
\tilde{f} : H/\alpha(H) \to \text{ker}\alpha_H
\]

\[
\tilde{x} \mapsto f(\overline{x})
\]

Let $\tilde{x} \in H/\alpha(H)$ then $x \in G$ and then $f(\overline{x}) \in \text{ker}\alpha$. But $\text{ker}\alpha$ is a subset of $H$ so $f(\overline{x}) \in H$ and the map $\tilde{f}$ is well defined.

We also have:

\[
\text{ker}\tilde{f} = \{ \tilde{x} \in H/\alpha(H) / f(\overline{x}) = 1 \} = \{ \tilde{x} \in H/\alpha(H) / x \in \alpha(G) \} = \{ x.\alpha(H) / x \in \alpha(G) \cap H \}. \quad \text{By our hypothesis} \ x \in \alpha(H) \quad \text{and then} \ x.\alpha(H) \subset \alpha(H). \quad \text{But the two sets are classes so they are equal and} \ \text{ker}\tilde{f} = \{ \alpha(H) \} \quad \text{and} \quad \text{the map} \ \tilde{f} \quad \text{is one-to-one and therefore is an isomorphism.}
\]

5 Groups of order $p^3$; $p$ an odd prime

Some results of this section are in [4] but the proofs are some times different. We also introduce the semidirect product to study this type of groups.

Let $G$ be a group of order $p^3$; $p$ an odd prime then $G$ belong to one of five isomorphic classes: $C_p^3, C_p \times C_p, (C_p)^3, (C_p \times C_p) \rtimes C_p$ and $(C_p^2) \rtimes C_p$. Let $\alpha \in \text{end}(G)$ be non trivial (not the endomorphism $x \mapsto 1; \forall x \in G$ nor an automorphism). First of all if $|\text{ker}\alpha| = p$; $|\alpha(G)| = p^2$ and then $\text{ker}\alpha$ and $G/\alpha(G)$ are cyclic of order $p$ and then isomorphic and $\alpha$ is morphic. We must study cases of $|\text{ker}\alpha| = p^2$; $|\alpha(G)| = p$ in each previous isomorphism classes.

1. The commutative case
(a) Case: $G = \mathbb{C}_{p^3}$

We know that $C_n$ is morphic for all integer $n$, so is $\mathbb{C}_{p^3}$.

In this case $ker\alpha$ and $\alpha(\mathbb{G})$ are cyclic of order $p^2$ and $p$ respectively. The are $p^2 - 1$ elements of $G = \langle a \rangle$ of order $p^2$ which are $a^{bp}; 1 \leq k \leq p^2 - 1$ and $p - 1$ elements of $G$ of order $p$ which are $a^{kp}; 1 \leq k \leq p - 1$. Because all subgroups considered and of the same order are isomorphic, we can take $\alpha(\mathbb{G}) = \langle a^{p^2} \rangle$ so $\mathbb{G}/\alpha(\mathbb{G}) = \langle a \rangle \simeq ker\alpha$ where $\overline{a} = \langle a^{p^2 + 1} \rangle$. $\alpha$ is then morphic.

(b) Case: $G = \mathbb{C}_{p^2} \times \mathbb{C}_p$.

Let

$$\alpha : \begin{array}{ccc}
G & \longrightarrow & G \\
(a^k, b^m) & \longmapsto & (1, b^m)
\end{array}$$

It is easy to see that $\alpha(\mathbb{G}) = 1 \times \mathbb{C}_p \simeq \langle (a^p, 1) \rangle$ such a subgroup of $\mathbb{G}$. We also have $ker\alpha = \mathbb{C}_{p^2} \times 1 \simeq \mathbb{C}_{p^2}$. Finally one have

$$\mathbb{G}/\alpha(\mathbb{G}) \simeq \mathbb{G}/\langle (a^p, 1) \rangle \simeq \mathbb{C}_p \times \mathbb{C}_p \not\simeq \mathbb{C}_{p^2}.$$ $\alpha$ isn’t morphic.

(c) Case: $G = (\mathbb{C}_p)^3$

If $ker\alpha \simeq \mathbb{C}_{p^2}$ then there exists one element $(x_1, x_2, x_3) \in \mathbb{G}$ such $ker\alpha = \langle (x_1, x_2, x_3) \rangle$ so the order of $(x_1, x_2, x_3)$ is $p^2$ which is impossible; (the element of $\mathbb{G}$ have an order less than $p$). So $ker\alpha$ must be isomorphic to $(\mathbb{C}_p)^2$. We can take $\alpha(\mathbb{G}) = \mathbb{C}_p \times 1 \times 1$ and $ker\alpha = 1 \times (\mathbb{C}_p)^2$; then the map

$$\Phi : \mathbb{G}/\alpha(\mathbb{G}) \longrightarrow ker\alpha; \overline{(1, b^k, c^m)} \longmapsto (1, b^k, c^m)$$

is an isomorphism so $\alpha$ is then morphic.

2. The noncommutative case

(a) Case: $G = (\mathbb{C}_{p^2}) \rtimes \mathbb{C}_p$

**Lemma 5.1** If $gcd(k, p) = 1$ then the morphism given by

$$f_p : \begin{array}{ccc}
\mathbb{C}_{p^2} & \longrightarrow & \mathbb{C}_{p^2} \\
b & \longmapsto & b^k
\end{array}$$

is an automorphism.

**Proof 5.2** The proof is easy since $gcd(k, p) = 1 \Longrightarrow gcd(k, p^2) = 1$ and then $b^{km} = 1$ and $gcd(k, p^2) = 1 \Longrightarrow p^2|m$ and finally $f_p$ is one-to-one.
Lemma 5.3 Let \( k \equiv 1[p] \) and \( \Psi \) be the group morphism defined by:

\[
\Psi : C_p \longrightarrow \text{Aut}(C_{p^2}) \quad a 
\mapsto f^p_p;
\]

we then have \( \Psi^p(a) = \text{id}_{C_{p^2}} \).

Proof 5.4 We have \( \Psi^p(a) = f^p_p \) so \( f^p_p(b) = b^{k^p} \) but \( k \equiv 1[p] \implies k^p \equiv 1[p^2] \) and \( b^{k^p} = b \). Finally \( f^p_p = \text{id}_{C_{p^2}} \).

Lemma 5.5 If \((b^{m_1}, a^{n_2}), (b^{n_1}, a^{n_2}) \in C_{p^2} \times C_p\) then the law of semidirect product \( C_{p^2} \rtimes C_p \) is defined by:

\[
(b^{m_1}, a^{n_2}), (b^{n_1}, a^{n_2}) = (b^{m_1+n_1k^{m_2}}, a^{m_2+n_2}),
\]

where the integer \( k \) is such \( k \equiv 1[p^2] \).

Proof 5.6 We know that if \( \Psi \) is such the conditions of the lemmas 5.1 and 5.2 are fulfil then the law of the semidirect product is such:

\[
(b^{m_1}, a^{n_2}), (b^{n_1}, a^{n_2}) = (b^{m_1}\Psi(a^{n_2})(b^{n_1}), a^{m_2+n_2}).
\]

But \( \Psi(a^{n_2}) = f^p_{n_2} \). So \( f^p_{n_2}(b^{n_1}) = (f^p_{n_2}(b))^{n_1} = (b^{k^{n_2}})^{n_1} \) and then

\[
(b^{m_1}\Psi(a^{n_2})(b^{n_1}), a^{m_2+n_2}) = (b^{m_1+n_1k^{m_2}}, a^{m_2+n_2}).
\]

Lemma 5.7 In the case \( p = 3 \) the only element of \( C_9 \rtimes C_3 \) are:

\[
(3, 0), (3, 1), (3, 2), (6, 0), (6, 1), (6, 2).
\]

The only normal subgroup of order 3 is \( < (3, 0) > \).

Lemma 5.8 Let \( \alpha \) be the endomorphism defined by:

\[
\alpha : G = C_9 \rtimes C_3 \longrightarrow C_9 \rtimes C_3 \quad \begin{array}{c}
(1, 0) \\
(0, 1)
\end{array} \mapsto \begin{array}{c}
(3, 0) \\
(0, 0)
\end{array} .
\]

Then \( \alpha(G) = < (3, 0) > < G \) and in \( G/\alpha(G) \);

\[
(k, m) = \{(k, m), (k + 3, m), (k + 6, m)\}.
\]

Proposition 5.9 The endomorphism \( \alpha \) of the lemma 5.5 is morphic and so \( G = C_9 \rtimes C_3 \) is morphic.
Proof 5.10 In this case we have \( \ker \alpha = \{(3k, m) / 0 \leq k \leq 2; 0 \leq m \leq 2\} \) and the map:

\[
\Gamma: \ker \alpha \longrightarrow G/\alpha(G) \\
(3k, m) \longrightarrow (k, m)
\]

is such:

\[
\Gamma((3k, m).(3k', m')) = \Gamma((3k + 3k'.4^m, m + m')) = (k + k'.4^m, m + m') =
\]

\[
= (k, m).(k', m') = \Gamma((3k, m)).\Gamma((3k', m'))
\]

and the map is a group morphism and also one-to-one, so it realizes an isomorphism and \( \alpha \) is then morphic.

By the lemma 5.4, we deduce that \( G \) is morphic.

(b) Case: \( G = (C_p \times C_p) \rtimes C_p \)

Lemma 5.11 Let \( f \) be the endomorphism defined by:

\[
f : C_p \times C_p = < a > \times < b > \longrightarrow C_p \times C_p \\
(a, 1) \longmapsto (a, 1); \\
(1, b) \longmapsto (a, b)
\]

then \( f \) is an automorphism of \( G \).

Lemma 5.12 Let \( f \) denotes the automorphism of the previous lemma and define the group morphism:

\[
\Psi: C_p = < c > \longrightarrow \text{Aut}(C_p \times C_p) \\
c \longmapsto f
\]

Then \( \Psi \) is such \( \Psi^p = \text{id}_{C_p \times C_p} \) and then defines a semidirect product of \( C_p \times C_p \) by \( C_p \). More precisely if \( (a^m, b^n, c^k), (a^{m'}, b^{n'}, c^{k'}) \) are elements of \( < a > \times < b > \times < c > \) then their product in \( (C_p \times C_p) \rtimes C_p \) is defined by:

\[
(a^m, b^n, c^k).(a^{m'}, b^{n'}, c^{k'}) = (a^{m+m'+kn'}, b^{n+n'}, c^{k+k'})
\]

Proof 5.13 \( \Psi \) is well defined and since every non trivial element of \( < c > \) is of order \( p \) we have \( \Psi^p(c) = \Psi(c^p) = \Psi(1) = \text{id}_{C_p \times C_p} \). In the other side we have \( f^p(a, b) = (a^{p+1}, b) = (a, b) \). These conditions define a semidirect product. More precisely if \( (a^m, b^n, c^k), (a^{m'}, b^{n'}, c^{k'}) \) are elements of \( < a > \times < b > \times < c > \) then

\[
(a^m, b^n, c^k).(a^{m'}, b^{n'}, c^{k'}) = ((a^m, b^n).f^k(a^{m'}, b^{n'}), c^{k+k'}) =
\]

\[
= ((a^m, b^n).(a^{m'+kn'}, b^{n'}), c^{k+k'}) = (a^{m+m'+kn'}, b^{n+n'}, c^{k+k'})
\]
Lemma 5.14 The group endomorphism $\alpha$ defined by:

$$
\alpha : G = (C_p \times C_p) \rtimes C_p \longrightarrow G
$$

$$(a, 1, 1) \longrightarrow (a, 1, 1)$$
$$(1, b, 1) \longrightarrow (1, 1, 1)$$
$$(1, 1, c) \longrightarrow (1, 1, 1)$$

is morphic.

Proof 5.15 One can easily see that $(a^m, b^n, c^k) = (a^m, 1, 1)(1, b^n, 1)(1, 1, c)$ so $\alpha((a^m, b^n, c^k)) = (a^m, 1, 1)(1, 1, 1)(1, 1, 1) = (a^m, 1, 1) = (a, 1, 1)$ and $\alpha(G) = \langle (a, 1, 1) \rangle = C_p \times 1 \times 1$. Note that the inverse of $(a^m, b^n, c^k)$ is the element $(a^{-m+k(n-p)}, b^{p-n}, c^{p-k})$ and then

$$(a^m, b^n, c^k)(a^{m'}, 1, 1)(a^{-m+k(n-p)}, b^{p-n}, c^{p-k}) =
(a^{m+m'}, b^n, c^k)(a^{-m+k(n-p)}, b^{p-n}, c^{p-k}) = (a^{m'}, 1, 1).$$

We conclude that $\alpha(G)$ is normal in $G$.

Let $f$ be the endomorphism of $G$ defined by:

$$
\begin{align*}
f : & \quad G \longrightarrow G \\
(a, 1, 1) & \longrightarrow (1, 1, 1) \\
(1, b, 1) & \longrightarrow (1, b, 1) \\
(1, 1, c) & \longrightarrow (1, 1, c)
\end{align*}
$$

We have $\ker f = C_p \times 1 \times 1 \simeq \alpha(G)$ and $f(G) = 1 \times C_p \times C_p \simeq \ker \alpha$ so

$$G/\alpha(G) \simeq G/\ker f \simeq f(G) \simeq \ker \alpha$$

and $\alpha$ is morphic.

Theorem 5.1 The group $C_p \times C_p \rtimes C_p$ is morphic.

Proof 5.16 Let $\alpha$ like in lemma 5.9 and $\beta$ be an other endomorphism of $G$. If $|\beta(G)| = p^2$, then it is obvious that $G/\beta(G) \simeq \ker \beta$. If $|\beta(G)| = p$, $\ker \beta$ is of order $p^2$ so it is isomorphic to $C_p \times C_p$ or $C_p^2$, but in the last case it must exists an element in $G$ of order $p^2$ which is impossible; then $\ker \beta \simeq C_p \times C_p \simeq 1 \times C_p \times C_p$, so $\ker \beta \simeq \ker \alpha$ $\simeq G/\alpha(G) \simeq G/\beta(G)$ and $\beta$ is morphic and finally $G$ is morphic.

References


Received: October, 2012