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On Generalised Pre Regular Weakly(gprw)-Closed and gprw-Quasi Closed Functions in Topological Spaces

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Abstract

The aim of this paper is to introduce and study *gprw*-closed and *gprw*-quasi closed functions in topological spaces. Also some properties of these classes of functions will be discussed here.

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1 Introduction

In 1970, Levine [6] initiated the study of so called generalized closed sets. This notion has been studied extensively in the recent years by many topologists because generalized closed sets are not only natural generalization of closed sets. More over, they also suggest several new properties of topological spaces. Most of these new properties are separation axioms weaker than T_1, some of them found to be useful in computer science and digital topology. Furthermore, the study of generalized closed sets also provides new characterization of some known classes of spaces. In 1997 Y. Gnanambal [4] proposed the definition of generalized pre regular-closed sets (briefly *gpr*-closed) and further notion of pre regular $T_{1/2}$ space and generalized pre regular continuity was introduced. And in 2007, notion of regular weakly closed set is defined by S.S. Benchalli and R.S. Wali [1] and they proved that this class lie between the class of all w-closed sets [11] and the class of all regular generalized closed sets [10] and some of its properties are investigated in it. Now in this paper, the notion of *gprw*-closed functions and *gprw*-quasi closed functions are defined by using the notions of *gprw*-closed sets which is introduced by S. Mishra, V. Joshi and N. Bhardwaj[8]. Also properties of these classes of functions are studied in this paper.

2 Preliminary Notes

Definition 2.1 *A subset A of X is called regular open (briefly r-open) [9] set if $A = int(cl(A))$ and regular closed (briefly r-closed) [9] set if $A = cl(int(A))$.*

Definition 2.2 *A subset A of X is called pre-open set [7] if $A \subseteq int(cl(A))$ and pre-closed [7] set if $cl(int(A)) \subseteq A$.*

Definition 2.3 *A subset A of a space (X, τ) is called regular semiopen [2] if there is a regular open set U such that $U \subset A \subset cl(U)$. The family of all regular semiopen sets of X is denoted by $RSO(X)$.*

Definition 2.4 *Let (X, τ) be a topological space and A be a subset of X said to be gprw-closed [8] if $pcl(A) \subset U$, whenever $A \subset U$ and U is regular semi open. The family of all gprw-closed set in space X is denoted by $GPRWC(X)$.*

Definition 2.5 *A subset of a topological space (X, τ) is called*

1. *Generalized pre regular closed (briefly gpr-closed) [4] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*

2. *π-generalized closed (briefly π-g-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open in X.*

3. *Weakly closed (briefly w-closed) [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in X.*

4. *Regular weakly closed (briefly rw-closed) [1] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen in X. We denote the set of all rw-closed sets in X by $RWC(X)$.*

Theorem 2.6 *Every regular semiopen set in X is semiopen but not conversely.*

Theorem 2.7 *[5] If A is regular semiopen in X, then $X \setminus A$ is also regular semiopen.*

3 gprw-closed and gprw-quasi closed Functions in Topological Spaces

Definition 3.1 *A function $f : (X, \tau) \to (Y, \sigma)$ on topological spaces is said to be $gprw$-closed if $f(V)$ is $gprw$-closed in (Y, σ) for every closed set V in (X, τ).*

Theorem 3.2 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $gprw$-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a $gprw$-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. Let U^c is closed set in Y. Then $f(U^c)$ is $gprw$-closed set in X, since f is $gprw$-closed. So $Y \setminus f(U^c)$ is $gprw$-open in X. Thus $V = Y \setminus f(U^c)$ is a $gprw$-open set containing S such that $f^{-1}(V) \subset U$.
Conversely, by hypothesis there is $gprw$-open set V of Y such that $Y \setminus V \subset f(V) \subset f(X \setminus f^{-1}(V)) \subset Y|V$ which implies $f(F) = Y \setminus V$. Since $Y \setminus S$ is $gprw$-closed $f(F)$ is $gprw$-closed and thus f is $gprw$-closed.

Theorem 3.3 *If a map $f : (X, \tau) \to (Y, \sigma)$ is closed and map $g : (Y, \sigma) \to (Z, \eta)$ is $gprw$-closed then $gof : (X, \tau) \to (Z, \eta)$ is $gprw$-closed map.*

Proof. Let H be a closed set in (X, τ). Then $f(H)$ is closed in (Y, σ), by the definition of closed map. Now $gof(H) = g(f(H))$ and by the definition of g we have $g(f(H)) = gof(H)$ is $gprw$-closed (Z, η). Thus from an arbitrary closed set H in (X, τ) and we have its image under gof is $gprw$-closed in (Z, η). Hence $gof : (X, \tau) \to (Z, \eta)$ is $gprw$-closed.

Definition 3.4 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be gprw-open if $f(V)$ is gprw-open in Y for every open set V in X.*

Definition 3.5 *A function $f : (X, \tau) \to (Y, \sigma)$ said to be quasi gprw-closed if the image of each gprw-closed set in X is closed in Y.*

Theorem 3.6 *Every quasi gprw-closed function is closed as well as gprw-closed function.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be quasi $gprw$-closed function on two topological spaces. Now let us consider any closed set F of (X, τ), By [8] F is $gprw$-closed set in (X, τ), also we are given that f is quasi $gprw$-closed therefore image of F is closed in (Y, σ) that shows that $f : (X, \tau) \to (Y, \sigma)$ is closed map. And to show that it is $gprw$-closed we consider a closed set V and then it is $gprw$-closed by [8], now its image under quasi $gprw$-closed function f is closed in (Y, σ), and it will be $gprw$-closed in Y. Therefore f is $gprw$-closed.

Remark 3.7 *Every gprw-closed function need not be quasi gprw-closed it can be seen by the following example.*

Example 3.8 *Let a topological spaces $X = Y = \{1, 2, 3\}$ with topologies $\tau = \{\phi, X, \{1, 2\}\}$ and $\sigma = \{\phi, Y, \{1\}, \{2, 3\}\}$ respectively. Now if we define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(1) = 2$,$f(2) = 3$,$f(3) = 1$. Then this function is gprw-closed but not gprw-quasi closed, since the image of gprw-closed set is not closed for example $\{1\}$ has image $\{2\}$ and $\{2, 3\}$ has image $\{1, 3\}$ which are not closed.*

Theorem 3.9 *A function $f : X \to Y$ said to be quasi gprw-closed if and only if for every subset U, we have $cl(U) \subset f(gprw - cl(U))$.*

Proof. As $U \subset gprw - cl(U)$, for an arbitrary set U, therefore by taking f on both sides we get $f(U) \subset f(gprw - cl(U))$. Now as $f(gprw - cl(U))$ is closed therefore $cl(f(U)) \subset cl(f(gprw - cl(U))) = f(gprw - cl(U))$. Hence $cl(U) \subset f(gprw - cl(U))$.
Conversely, let U is a $gprw$-closed set in X. Then $f(U) = f(gprw - cl(U)) \supset cl(f(U))$, by hypothesis therefore we have $f(U) \supset cl(f(U))$, but we know for any set U, $f(U) \subset cl(f(U))$ always therefore we have $f(U) = cl(f(U))$. Hence $f(U)$ is closed, so f is quasi $gprw$-closed function.

Theorem 3.10 *If $f : X \to Y$ is quasi gprw-closed function, then $f^{-1}(cl(B)) \subset gprw - cl(f^{-1}(B))$ for every subset B of Y*

Proof. Let B be an arbitrary subset of Y then $gprw - cl(f^{-1}(B))$ is $gprw$-closed in X and f is quasi $gprw$-closed. Now $cl(B) \subset cl(f(f^{-1}(B))) \subset f(gprw - cl(f^{-1}(B)))$. Thus $f^{-1}(cl(B)) \subset gprw - cl(f^{-1}(B))$.

Theorem 3.11 *A function $f : X \to Y$ is quasi gprw-closed if and only if for any subset B of Y and for any gprw-open set G of X containing $f^{-1}(B)$, then there exist an open set U of Y containing B such that $f^{-1}(U) \subset G$.*

Proof. Let $F = Y \setminus f(X \setminus G)$ then It is clear that $f^{-1}(B) \subset G$ implies $B \subset F$. Since f is quasi $gprw$-closed, we obtain F as a open set of Y moreover, we have $f^{-1}(U) \subset G$
Conversely, let F be a $gprw$-closed set of X and put $B = Y \setminus f(F)$. Then $X \setminus F$ is $gprw$-open set in X, containing $f^{-1}(B)$. By hypothesis, there exist a open set U of Y such that $B \subset U$ and $f^{-1}(U) \subset X \setminus F$. Hence we obtained $f(F) \subset Y \setminus U$. On the other hand, it follows that $B \subset U$, $Y \setminus U \subset Y \setminus B = f(B)$. Thus we obtain $f(F) = Y \setminus U$ which is closed and hence f is quasi $gprw$-closed function.

Theorem 3.12 *If $f : X \to Y$ and $g : Y \to Z$ are two quasi gprw-closed function then $gof : X \to Z$ is quasi gprw-closed function.*

Proof. Let B any $gprw$-closed set in X, Now as $f : X \to Y$ is $gprw$-closed function therefore $f(B)$ is closed in Y. Now by [8], it is $gprw$-closed further $g : Y \to Z$ is also $gprw$-closed function, so $g(f(B))$ is $gprw$-closed set in topological space Z. Hence we consider an arbitrary $gprw$-closed set in X and got its image closed under gof therefore $gof : X \to Z$ is quasi $gprw$-closed function.

Definition 3.13 *A function $f : X \to Y$ is called gprw**-closed function if the image of each gprw-closed subset of X is gprw-closed in Y.*

Theorem 3.14 *Let $f : X \to Y$ and $g : Y \to Z$ be two functions on topological spaces then*

1. *If f is gprw-closed and g is quasi gprw-closed then $gof : X \to Z$ is closed function.*
2. *If f is quasi gprw-closed and g is gprw- closed then $gof : X \to Z$ is gprw**- closed function.*
3. *If f is gprw**- closed and g is quasi gprw-closed then $gof : X \to Z$ is quasi gprw-closed function.*

Proof.

1. Let closed subset F of topological space X, then by the given fact that f is $gprw$-closed function, $f(F)$ is $gprw$-closed set in Y. Since g is quasi $gprw$-closed function from Y to Z therefore $g(f(F))$ is closed in Z. Hence image of an arbitrary closed set in X under gof is closed in Z, so $gof : X \to Z$ is closed function.

2. Let us consider *gprw*-closed set F of topological space X then by definition of f which is *gprw*-closed we have $f(F)$ is closed set in Y. Since g is *gprw*-closed function therefore $g(f(F)) = gof(F)$ is *gprw*-closed set in Z. Hence $gof : X \to Z$ is $gprw^{**}$-closed function.

3. Consider any *gprw*-closed set F in X then by definition of f which is $gprw^{**}$-closed function, the set $f(F)$ is *gprw*-closed in Y. Now we are given $g : Y \to Z$ is quasi *gprw*-closed set, therefore $g(f(F)) = gof(F)$ is closed set of Z. Now if we take a *gprw*-closed in X then its image under gof is closed set in Z. Hence $gof : X \to Z$ is quasi *gprw*-closed function.

Theorem 3.15 *Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $gof : X \to Z$ is quasi gprw-closed function on topological spaces then*

1. *If f is gprw-irresolute and surjective function then g is closed function.*

2. *If g is gprw-continuous and injective function, then f is $gprw^{**}$-closed function.*

Proof.

1. Suppose that F is an arbitrary closed set in Y. As f is *gprw*-irresolute therefore $f^{-1}(F)$ is *gprw*-closed in X. Since gof is quasi *gprw*-closed and f is surjective therefore $gof(f^{-1}(F)) = g(F)$ which is closed in Z. This implies g is closed function.

2. Suppose that F be any arbitrary *gprw*-closed set in X. Since gof is quasi *gprw*-closed therefore $gof(F)$ is closed set in Z. And also g is *gprw*-continuous injective function therefore $g^{-1}(gof(F)) = g^{-1}(g(f(F))) = f(F)$, which *gprw*-closed in Y. Hence f is $gprw^{**}$-closed function.

Theorem 3.16 *The function $g : X \to Y$ is quasi gprw-closed if and only if $g(X)$ is closed in Y and $g(V) \setminus g(X \setminus V)$ is open in $g(X)$ whenever V is gprw-open in X.*

Proof. Since X is *gprw*-closed therefore $g(X)$ is closed in Y and $g(V) \setminus g(X \setminus V) = g(V) \cap g(X) \setminus g(X \setminus V)$ is open in $g(X)$ when V is *gprw*-open in X. Conversely suppose $g(X)$ is closed in Y, $g(V) \setminus g(X \setminus V)$ is open in $g(X)$ when V is *gprw*-open in X, and let C be closed in X. Then $g(C) = g(X) \setminus g(X \setminus C) \setminus g(C)$ is closed in $g(X)$ and hence closed in Y.

Corollary 3.17 *Let X and Y be topological space. Then a surjective function $g : X \to Y$ is quasi gprw-closed if and only if $g(V) \setminus g(X \setminus V)$ is open in Y whenever U is gprw-open.*

Corollary 3.18 *Let $g : X \to Y$ be a gprw-continuous, quasi gprw-closed and surjective function then the topology on Y is $\{\ g(V) \setminus g(X \setminus V)$: V is gprw-open in X $\}$.*

Proof. Let W be open in Y. Then $g^{-1}(W)$ is *gprw*-open in and $g(g^{-1}(W)) \setminus g(X \setminus g^{-1})(W)) = W$. Hence all open set in Y are of the form $g(V) \setminus g(X \setminus V)$, V is *gprw*-open in X. On the other hand, all sets of the form $g(V) \setminus g(X \setminus V)$, V is *gprw*-open in X are open in Y from the above corollary.

Definition 3.19 *A function $f : X \to Y$ from a topological space X to a topological space Y said to be quasi gprw-open if the image of every gprw-open set in X is open in Y.*

Theorem 3.20 *A function $f : X \to Y$ is said to be quasi gprw-open if and only if for every subset U of X, $f(gprw - int(U)) \subset int(f(U))$.*

Proof. As $gprw - int(U) \subset U$, for an arbitrary set U, then we have $f(gprw - int(U)) \subset f(U)$. Now as $f(gprw - int(U))$ is open in Y therefore $int(f(U)) \supset int(f(gprw - int(U))) = f(gprw - int(U))$. Hence $f(gprw - int(U)) \subset int(f(U))$.
Conversely, assume that U is a *gprw*-open set in X. Then $f(U) = f(gprw - int(U)) \subset int(f(U))$ by hypothesis therefore we have $f(U) \subset int(f(U))$, but $int(f(U)) \subset f(U)$ always therefore we have $f(U) = int(f(U))$. Hence $f(U)$ is open, so that f is quasi *gprw*-open function.

Theorem 3.21 *If $f : X \to Y$ is quasi gprw-open function then $gprw - int(f^{-1}(G)) \subset f^{-1}(int(G))$ for every subset G of Y.*

Proof. Let G be an arbitrary sub set of Y then $gprw - int(f^{-1}(G))$ is a *gprw*-open set in X and f is quasi *gprw*-open, then $f(gprw - int(f^{-1}(G))) \subset int(f(f^{-1}(G))) \subset int(G)$. Hence $gprw - int(f^{-1}(G)) \subset f^{-1}(int(G))$ for an arbitrary subset G of Y.

Theorem 3.22 *For a function $f : X \to Y$ the following statements are equivalent*

1. *f is quasi gprw-open.*

2. *For each subset U of X, $f(gprw - int(U)) \subset int(f(U))$.*

3. *For each $x \in X$ and each gprw-neighbourhood U of $x \in X$, there exist a neighbourhood V of $f(x)$ in Y such that $V \subset f(U)$.*

Proof. (1) $\Rightarrow$ (2) it follows from the theorem 3.21.
(2) $\Rightarrow$ (3) Let $x \in X$ and U be an arbitrary $gprw$-neighbourhood of x in X. Then there exist a $gprw$-open set V in X such that $x \in V \subset U$. Then by (2) we have $f(V) = f(gprw - int(V)) \subset int(f(V))$ and as for an arbitrary set we have $int(f(V)) \subset f(V)$, therefore we arrive at $f(V) = int(f(V))$. So $f(V)$ is open in Y such that $f(x) \in f(V) \subset f(U)$. Hence for each $x \in X$ and each $gprw$-neighbourhood U of x there exist a neighbourhood $f(V)$ of $f(x)$ in Y such that $V \subset f(U)$.
(3) $\Rightarrow$ (1) Let U be an arbitrary $gprw$-open set X. Now for each $y \subset f(U)$ by (3) there exist neighbourhood V_Y of y in Y such that $V_Y \subset f(U)$. As V_Y is neighbourhood of y, that is why there exist an open set W_Y in Y such that $y \in W_Y \subset V_Y$. Thus $f(U) = \bigcup\{W_Y : y \in f(U)\}$ which is open set in Y. This shows that f is quasi $gprw$-open function.

Theorem 3.23 *A function $f : X \to Y$ is quasi gprw-open if and only if for any subset B of Y and for any gprw-closed set F of X, containing $f^{-1}(B)$. Then there exist a closed set G of Y containing B such that $f^{-1}(G) \subset F$.*

Proof. Let $B \subset Y$ and F be a $gprw$-closed set of X containing $f^{-1}(B)$. Now take $G = Y \setminus f(X \setminus F)$, it is clear that $f^{-1}(B) \subset F$ implies $B \subset G$. By hypothesis on f, G is closed set of Y. Moreover we have $f^{-1}(G) \subset F$.
Conversely, let U be a $gprw$-open set of X and put $B = Y \setminus f(U)$. Then $X \setminus U$ is a $gprw$-closed set in Y containing $f^{-1}(B)$. By hypothesis there exist a closed set F of Y such that $B \subset F$ and $f^{-1}(F) \subset X \setminus U$. Hence we obtain $f(U) \subset Y \setminus F$. On the other hand, it follows that $B \subset F$, $Y \setminus F \subset Y \setminus B = f(U)$. Thus we got $f(U) = Y \setminus F$ which is open and hence f is quasi $gprw$-open function.

Theorem 3.24 *A function $f : X \to Y$ is quasi gprw-open if and only if $f^{-1}(cl(B)) \subset gprw - cl(f^{-1}(B))$ for every subset B of Y.*

Proof. For any subset B of Y, $f^{-1}(B) \subset gprw - cl(f^{-1}(B))$. Therefore by the above theorem there exists a closed set F in Y such that $B \subset F$ and $f^{-1}(B) \subset gprw - cl(f^{-1}(B))$. Therefore we obtained $f^{-1}(cl(B)) \subset f^{-1}(F) \subset gprw - cl(f^{-1}(B))$.
Conversely let $B \subset Y$ and F be a $gprw$-closed set of X containing $(f^{-1}(B))$. Put $W = cl_Y(B)$, then we have $B \subset W$ and W is closed set and $f^{-1}(W) \subset gprw - cl(f^{-1}(B)) \subset F$. Then by theorem above f is quasi $gprw$-open.

Theorem 3.25 *Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $gof : X \to Z$ is quasi gprw-open function and g is continuous injective then f is quasi gprw-open.*

Proof. For $gprw$-open set U in X, we have $gof(U)$ is open in Z. Since gof is quasi $gprw$-open and g is an injective continuous function, $f(U) = g^{-1}(gof(U))$ is open in Y. Hence f is quasi $gprw$-open.

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