A Note on the Order Bidual of f-Algebras

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Abstract

The paper deals with the Arens Multiplication which we accomplished in four steps in the order bidual X^{**} . It is shown that if f is an element of order dual X^* of X with $\varepsilon(f) \neq 0$ and $x \in X^+$, then $f \cdot x = 0$ implies $f(x) = 0$.

Mathematics Subject Classification : 06F25 , 47B65

Keywords: Arens multiplication , f-algebra , Dedekind complete , orthomorphism , order dual.

Introduction

A Riesz space E under an associative multiplication is said to be a Riesz algebra whenever the multiplication makes E an algebra (with the usual properties), and in addition it satisfies the following property : If $x, y \in E^+$, then $xy \in E^+$. A Riesz algebra E is said to be an f -algebra whenever $x \wedge y = 0$ implies $(xz) \wedge y = 0$ for each $z \in E^+$. An order bounded band preserving operator is known as an orthomorphism and the set of all orthomorphism on X is denoted by $Orth(X)$, [2] .

A subset A of Riesz space is said to be bounded from above whenever there exists some x satisfying $y \leq x$ for all $y \in A$. Similarly, a set A is said to be bounded from below whenever there exists some x such that $x \leq y$ holds for all $y \in A$. Finally, a set A is called order bounded if it is bounded both from above and below. An operator $T : E \rightarrow F$ that maps order bounded subsets of E onto order bounded subsets of F is called order bounded. An operator $T : E \rightarrow E$ on a Riesz space is said to be band preserving whenever T leaves all bands of E invariant, i.e., whenever $T(B) \subseteq B$ holds for each band B of E .

Let E be a Riesz space. A linear functional $f : E \rightarrow R$ is called order bounded if f maps order bounded subsets of E onto bounded subsets of R , [4]. The vector space $E^\sim$ of all order bounded linear functionals on E is called the order dual of E , i.e., $E^\sim := L_b(E, R)$. Let X be an Archimedean f -algebra with order dual $X^\sim$. Following construction [3,5,6,7] a multiplication can be introduced in the order bidual $X^{\sim\sim}$ of X . This is accomplished in four steps as explained below.

$$1) \text{ } Orth(X) \times X \rightarrow X$$

$$(T, x) \rightarrow T.x = T(x) \quad \text{for } T \in Orth(X), x \in X.$$

$$2) X \times X^\sim \rightarrow Orth(X)^\sim$$

$$(x, x') \rightarrow (x.x')T = x'(Tx) \quad \text{for } x' \in X^\sim, x \in X, T \in Orth(X).$$

$$3) Orth(X)^{\sim\sim} \times X^\sim \rightarrow X^\sim$$

$$(T, x') \rightarrow (T.x')x = T(x.x') \quad \text{for } T \in Orth(X)^{\sim\sim}, x' \in X^\sim, x \in X.$$

$$4) Orth(X)^{\sim\sim} \times X^{\sim\sim} \rightarrow X^{\sim\sim}$$

$$(T, \hat{x}) \rightarrow (T.\hat{x})(x') = \hat{x}(T.x') \quad \text{for } T \in Orth(X)^{\sim\sim}, x' \in X^{\sim\sim}.$$

Proposition 1 : Let $\alpha : Orth(X)^{\sim\sim} \rightarrow Orth(X^\sim)$ be a mapping defined by $\alpha(T)x' = Tx'$ for $T \in Orth(X)^{\sim\sim}$, $x' \in X^\sim$. Then α is a one-one, onto and algebra homomorphism.

Proof : We will prove the following, respectively.

- i) α is a linear mapping.
- ii) α is an one-one mapping.
- iii) α is an algebra homomorphism.

i) $\alpha : \text{Orth}(X)^{\sim\sim} \rightarrow \text{Orth}(X^{\sim})$, $\alpha(T)x' = T.x'$, for $T \in \text{Orth}(X)^{\sim\sim}$, $x' \in X^{\sim}$.

$$\alpha(T)x' = T.x'$$

α is a linear mapping :

a) $\alpha(T + S) = \alpha(T) + \alpha(S)$ we must show that $\alpha(T + S)x' = \alpha(T)x' + \alpha(S)x'$ is true for all $x \in X$.

$$[\alpha(T + S)x']_x = [(T + S).x']_x = (T + S)(x.x') \quad \text{From third product we have}$$

$$\begin{aligned} (T.x')_x &= T(x.x') \quad \text{and} \\ (T + S)(x.x') &= T(x.x') + S(x.x') \\ &= (T.x')_x + (S.x')_x \\ &= [\alpha(T)x']_x + [\alpha(S)x']_x. \end{aligned}$$

b) $\alpha(\lambda T) = \lambda \alpha(T)$ we must show that $\alpha(\lambda T)x' = \lambda \alpha(T)x'$ is true for all $x \in X$.

$$\begin{aligned} ([\alpha(\lambda T).x'])_x &= [(\lambda T).x']_x, \quad x' \in X^{\sim} \\ &= \lambda T(x.x') \\ &= \lambda (T.x')_x \\ &= \lambda [\alpha(T).x']_x. \end{aligned} \quad \text{So, } \alpha \text{ is a linear mapping.}$$

ii) α is one-one [i.e. $T \neq 0 \Rightarrow \alpha(T) \neq 0$]

$0 \neq T \in \text{Orth}(X)^{\sim\sim}$ there is a $\exists y \in \text{Orth}(X)^{\sim}$

Let us take as $y = x.x' \in \text{Orth}(X)^{\sim}$

$$Ty \neq 0$$

$$T(x.x') \neq 0$$

$$(T.x')_x = 0$$

$$[\alpha(T)x']_x \neq 0 \Rightarrow \alpha(T) \neq 0.$$

iii) α is an algebra homomorphism, to prove this claim,

$$\begin{aligned}
 \alpha(T.S) &= \alpha(T)\alpha(S) \quad \text{for all } x \in X. \\
 [\alpha(T.S).x']_x &= (T : S)(x.x') \quad , \quad x.x' \in Orth(X)^{\sim} \\
 &= T(S(x.x')) \\
 &= T(\alpha(S).x')_x \\
 &= T(S.x')_x \quad , \quad S.x' \in X^{\sim} \\
 &= [T(S.x')]_x \\
 &= \alpha(T)\alpha(S).
 \end{aligned}$$

Lemma 2[1,5] : Let $x \in X$, $f \in X^{\sim}$.

If the mapping $f.x : Orth(X) \rightarrow R$ is defined by $(f.x)(\pi) = (f \circ \pi).x$, for $\pi \in Orth(X)$ then $f.x \in OrthX^{\sim}$.

Proof : Consider the mapping (2)

$$\begin{aligned}
 X \times X^{\sim} &\rightarrow OrthX^{\sim} \\
 (x, f) &\rightarrow x \circ f \quad , \quad x \circ f \in OrthX^{\sim} \quad , \quad x \circ f \in Orth(X) \rightarrow R \quad \text{Then, we obtain} \\
 (x.f)(\pi) &= (f \circ \pi).x. \\
 \text{Let } \beta : Orth(X) &\rightarrow L_b(X) \text{ be a mapping defined by} \\
 T &\rightarrow \beta(T)x = Tx.
 \end{aligned}$$

Next, we obtain the following equalities:

$$\begin{aligned}
 (xf)(T) &= (Tx)(f) \quad , \quad f \in X^{\sim} \\
 &= \beta(T)x \\
 (x.f)(\pi) &= (\pi x)(f) \\
 &= (\beta(\pi)x).f \\
 &= f(\beta(\pi)x) \\
 &= f(\pi x). \quad \text{So,} \\
 (x.f)(\pi) &= f(\pi x) \text{ and } xf \in OrthX^{\sim}.
 \end{aligned}$$

Lemma 3 : Suppose the mapping $\phi : X^{\sim\sim} \rightarrow Orth(X)^{\sim\sim}$ is defined by

$\phi(F)(f) = F(f_x)$, for $F \in X$, $f \in \text{Orth}(X)^\sim$ and the mapping

$\alpha: \text{Orth}(X)^{\sim\sim} \rightarrow \text{Orth}(X^\sim)$ then

$\beta = \alpha \circ \phi$ is a algebra homomorphism.

$$\begin{array}{ccc}
 & \phi & \\
 & \rightarrow & \text{Orth}(X)^{\sim\sim} \\
 X^{\sim\sim} & & \downarrow \alpha \\
 & \rightarrow & \text{Orth}(X^\sim) \\
 \beta = \alpha \circ \phi & &
 \end{array}$$

Proof : We must show

$$\begin{aligned}
 \beta(F.G) &= \beta(F).\beta(G) \quad \text{for } F, G \in X^{\sim\sim} \\
 \beta(F.G) &= (\alpha \circ \phi)(F.G) \\
 &= \alpha(\phi(F.G)) \quad \text{we know } \phi \text{ is homomorphism [1]} \\
 &= \alpha(\phi(F).\phi(G)) \\
 &= \alpha(\phi(F).\alpha(\phi(G))) \quad (X^\sim \text{ is } f\text{-module over } \text{Orth}(X)^{\sim\sim}) \\
 &= (\alpha \circ \phi)(F)(\alpha \circ \phi)(G) \\
 &= \beta(F)\beta(G).
 \end{aligned}$$

For every $f \in (X^\sim)^+$, defined $\mathcal{E}(f)$ to be the set of all extensions of f in $(\text{Orth}(X)^\sim)^+$, [1]. That is , $\mathcal{E}(f) = \{g \in \text{Orth}(X)^\sim : 0 \leq g \text{ and } g_x = f\}$.

Proposition 4 : Let f be an element of order dual $X^\sim$ of X with $\mathcal{E}(f) \neq \emptyset$. If $x \in X^+$, then $f.x = 0$ implies $f(x) = 0$, [1].

Proof : Let us consider the set ,

$$\mathcal{E}(f) = \{g \in \text{Orth}(X)^\sim : 0 \leq g \text{ and } g_x = f\} \quad \text{for } f \in X^\sim , x \in X .$$

$$\begin{aligned}
 \text{Let } 0 &= f.x \in \text{Orth}(X^\sim) \\
 0(T) &= f(T.x) , \quad \forall T \in \text{Orth}(X) \\
 0 &= f(Tx)
 \end{aligned}$$

If we take $T = I$

$$f(Ix) = 0 \Rightarrow f(x) = 0.$$

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Received: September, 2012