Extended Results on f-Orthomorphisms over the Second Order Dual of an f-algebra

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Abstract

Let A be an Archimedean f-algebra with unit and L, M be two f-modules over A . In this paper we show that L'' is an f-module over A'' and A'' is topologically full over A'' . Assume that T is an f-orthomorphism from L to M over A . Also, we show that the second adjoint T'' of T is an f-orthomorphism from L'' to M'' over A'' , [12].

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1 Introduction

Let L and M be two Archimedean Riesz spaces and A be an f-algebra. The positive cone of L will be denoted by L_+ . A linear operator between two Riesz spaces is said to be order bounded if it maps order bounded subsets of L to order bounded subsets of M . The vector spaces of all order bounded operators

from the Riesz space L into Riesz space M will be denoted by $L_b(L, M)$. An operator $T : L \rightarrow L$ on a Riesz space L is said to be band preserving whenever T leaves all bands of L invariant, i.e., whenever $T(B) \subseteq B$ holds for each band B of L . Recall that in a Riesz space, two elements x and y are said to be disjoint (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. An operator $T : L \rightarrow M$ between two Riesz spaces is said to preserve disjointness whenever $x \perp y$ in L implies $Tx \perp Ty$ in M . $\pi \in L_b(L)$ is called an orthomorphism of L if $x \perp y$ in L imply that $\pi(x) \perp y$. $Orth(L)$ is a space of all order bounded and band preserving operators on L and the set of orthomorphisms on L is $Orth(L) = \{T \in L_b(L) : x \perp y \Rightarrow Tx \perp y\}$. As I is the identity operator in L , the principle order ideal generated by the identity operator I in $Orth(L)$ is called the ideal center of L and is denoted by $Z(L)$, [1, 3]. If L is a Dedekind complete Riesz space, $Orth(L) = B(I)$ as $B(I)$ is the band generated by I , [13]. The vector space L' of all order bounded linear functionals on L is called the order dual of L , i.e., $L' = L_b(L, R)$. L' is a Riesz space, [2, 14]. The order bidual L'' of L is the order dual of L' that is $L'' = (L')'$ and the order continuous part of the order bidual of L is denoted by $(L')'_n$. Recall that the expression L' separates the points of L means that for each $0 < x \neq 0$ there exists some $0 < f \in L'$ with $f(x) \neq 0$. The Riesz space A is said to be Riesz algebra (lattice ordered algebra) if there exists an associative multiplication in A with the usual algebra properties such that $ab \in A_+$ for all $a, b \in A_+$. Such a Riesz algebra A is called an f -algebra if A has the additional property that $a \perp b$ in A implies $ca \perp b$ and $ac \perp b$ for all $c \in A$ (in other words, left and right multiplication by c are orthomorphisms of A). Since every Archimedean f -algebra is commutative [7], we deal only with commutative f -algebras in this work. Let A be an f -algebra with A' separates the points of A . Let us recall also that $(A')'_n$, the order continuous part of the order bidual of an f -algebra A , [8]. If A has a multiplicative unit, then $(A')'_n = (A')'$, the whole order bidual of A , [8]. For unexplained notions, we refer to the books [2, 9, 10, 11, 14]. Here all Riesz spaces have separating order duals.

Definition 1 [12, 6] *Let A be an f -algebra with unit e and L be a Riesz space. L is said to be an f -module over A if there exists a map;*

$A \times L \rightarrow L : (a, x) \rightarrow a \cdot x$ satisfying,

- 1) L is a module over A and $e \cdot x = x$ for each $x \in L$
- 2) for each $a \in A_+$ and $x \in L_+$ we have $a \cdot x \in L_+$
- 3) If $x \perp y$ in L , then for each $a \in A$ we have $a \cdot x \perp y$.

Lemma 2 [12] *Let A be an f -algebra with unit e and L be a Riesz space. For all $x \in L$, $f \in L'$ and $a \in A$. Riesz space L' with $A \times L' \rightarrow L' : (a, f) \rightarrow a \cdot f : a \cdot f(x) = f(a \cdot x)$ is an f -module over A .*

In this case, we define a map $m : A \rightarrow \text{Orth}(L')$ defined by $m(a) = \pi'_\alpha$ where $\pi'_\alpha(f) = a \cdot f$ for each $f \in L'$. m is a unital algebra and Riesz homomorphism. This map induces an f-module structure on L' over A . If L is an f-module over A , we can define the mappings,[4],

$$1) L \times L' \rightarrow A'$$

$$(x, f) \rightarrow x \cdot f : (xf)(a) = f(a \cdot x) \text{ for } x \in L, f \in L', a \in A.$$

$$2) A'' \times L' \rightarrow L'$$

$$(F, f) \rightarrow F \cdot f : (F \cdot f)(x) = F(f \cdot x) \text{ for } x \in L, f \in L', F \in A''.$$

$$3) A'' \times L'' \rightarrow L''$$

$$(F, \hat{f}) \rightarrow (F \cdot \hat{f})(f) = \hat{f}(F \cdot f) \text{ for } f \in L', F \in A'', \hat{f} \in L''.$$

We introduce the mappings :

$$1) A \times L' \rightarrow L'$$

$$(a, f) \rightarrow a \cdot f : af(x) = f(ax) \text{ for } x \in L, f \in L', a \in A$$

$$2) L'' \times L' \rightarrow A'$$

$$(\hat{f}, f) \rightarrow \psi_{f, \hat{f}} : \psi_{f, \hat{f}}(a) = \hat{f}(a \cdot f) \text{ for } a \in A, f \in L', \hat{f} \in L''.$$

Theorem 3 [12] Suppose that L is an f-module over A .

Let $u : A \rightarrow \text{Orth}(L'')$ where $u(a)\hat{f} = a \cdot \hat{f}$ for each $\hat{f} \in L''$. $v : A'' \rightarrow \text{Orth}(L'')$ where $v_F(\hat{f}) = F \cdot \hat{f}$ for each $\hat{f} \in L''$. Then u and v are positive operators with $u(A) \subseteq \text{Orth}(L'')$, $v(A'') \subseteq \text{Orth}(L'')$.

Also, u and v are unital algebra and Riesz homomorphism.

Proof: Here u and v are positive operators.

Let $\pi_\alpha \in \text{Orth}(L)$ and we denote the adjoint of π_α by π'_α .

$m : A \rightarrow \text{Orth}(L') : a \rightarrow m(a) = \pi'_\alpha$ where $\pi'_\alpha(f) = a \cdot f$ for $f \in L'$ and

$k : \text{Orth}(L') \rightarrow \text{Orth}(L'') : \pi' \rightarrow k(\pi') = \pi''$ are Riesz and algebra homomorphisms. m is a unital algebra and Riesz homomorphism by Proposition 2.2[12].

The mapping k is injective, positive and satisfies $k(I') = I''$ where I'' denotes the identity mapping of L'' . $\text{Orth}(L'')$ is commutative so,

$k(\pi'_1 \pi'_2) = (\pi'_1 \pi'_2)' = \pi''_2 \pi''_1 = \pi''_1 \pi''_2 = k(\pi_1)k(\pi_2)$ for all $\pi'_1, \pi'_2 \in \text{Orth}(L')$. By Corollary 5.5 of [7] k is an Riesz homomorphism.

For $a \in A$, $\hat{f} \in L''$, and $f \in L$ we have $u(a)(\hat{f})(f) = a \cdot \hat{f}(f) = \hat{f}(af) = \hat{f}(\pi'_\alpha(f)) = \pi''_\alpha(\hat{f})(f) = (k \circ m)(a)(\hat{f})(f)$. k and m are algebra homomorphisms since $u(a \cdot b) = (k \circ m)(a \cdot b) = k(m(a \cdot b)) = k(m(a) \cdot m(b)) = k(m(a)) \cdot k(m(b)) = (k \circ m)(a)(k \circ m)(b) = u(a) \cdot u(b)$. Thus, u is an algebra homomorphism. Let $\hat{a}$ be the image in A'' and for $a \in A$, $\hat{f} \in L''$, $f \in L'$, $v_{\hat{a}}(\hat{f}) = \hat{a} \circ \hat{f}(f) = \hat{a}(\psi_{f, \hat{f}})(a) = \hat{f}(a \cdot f) = a \cdot \hat{f}(f) = u(a)\hat{f}(f)$, and $v_{\hat{a}}(\hat{f}) \in \text{Orth}(L'')$.

Let $I(A)$ denote the ideal generated by A in A'' and $0 \leq F \in I(A)$. Hence there exists $a \in A$ with $0 \leq F \leq \hat{a}$. As $v : A'' \rightarrow \text{Orth}(L'')$ is positive we have $0 \leq v_F \leq v_{\hat{a}}$ then $v_F \in \text{Orth}(L'')$. We choose G_α in $I(A)$ with $G_\alpha \uparrow F$. As $0 \leq \psi_{f, \hat{f}} \in A'$ for each $f \in L'_+$ and $\hat{f}_+ \in L''$, we have

$G_\alpha(\psi_{f, \hat{f}}) \uparrow F(\psi_{f, \hat{f}})$ so $G_\alpha \circ \hat{f}(f) \uparrow F \circ \hat{f}(f)$.

$G_\alpha \circ \hat{f} \uparrow F \circ \hat{f}$ and $v_{G_\alpha} \uparrow v_F$.

Since $v_{G_\alpha} \in \text{Orth}(L'')$ and $\text{Orth}(L'')$ is a band we get $v_F \in \text{Orth}(L'')$.

Consider the mapping $v : A'' \rightarrow \text{Orth}(L'')$ defined by $v(F)(\hat{f}) = F \cdot \hat{f} = v_F(\hat{f})$ for all $F \in A''$ and $\hat{f} \in L''$,

$((F \cdot G) \cdot \hat{f})(f) = (F \cdot G)(\hat{f} \cdot f) = F(G \cdot (\hat{f} \cdot f)) = F((G \cdot \hat{f}) \cdot f) = (F \cdot (G \cdot \hat{f}))(f)$ for all $f \in L'$ and $\hat{f} \in L''$.

Therefore, $v_{F \cdot G} = v_F \cdot v_G$ for all $F \cdot G \in A''$. v is an algebra homomorphism.

Suppose that $F \wedge G = 0$ in A'' . Then $v(F \cdot G) = v(F)v(G) = 0$ in $\text{Orth}(L'')$ as $FG = 0$. However, $\text{Orth}(L'')$ is a unital and hence semiprime f -algebra, [5, 8] so $v(F) \wedge v(G) = 0$. We may conclude that v is disjoint preserving and from [7] it is also Riesz homomorphism.

Theorem 4 [12] *Let A be an f -algebra with unit and L be an f -module over A . Then Riesz space L'' is an f -module over A'' .*

Proof: From the consequence of previous theorem, L'' is an f -module over A'' .

That is,

$$A'' \times L'' \rightarrow L'' : (F, \hat{f}) \rightarrow (F \cdot \hat{f})(f) = \hat{f}(F \cdot f)$$

satisfies the following conditions;

- 1) $F \cdot (G \cdot \hat{f}) = (F \cdot G)\hat{f}$, for all $F, G \in A''$ and $\hat{f} \in L''$
- 2) $0 \leq F \in A''$, $0 \leq \hat{f} \in L''$, $F \cdot \hat{f} \geq 0$
- 3) If $\hat{f} \perp \hat{g}$ in L'' , then $F \cdot \hat{f} \perp \hat{g}$ in L'' , for all $\hat{f}, \hat{g} \in L''$ and $F \in A''$.

The above theorem shows that, when A'' is an f -algebra with unit, L'' is a unital f -module over A'' and multiplication satisfies $E \cdot \hat{f} = \hat{f}$ for all $\hat{f} \in L''$ and $E \in A''$ (E is the unit element in A'').

Definition 5 [12] *The f -module L'' over A'' with unit E is said to be topologically full with respect to A'' if for two arbitrary vectors $\hat{f}, \hat{g}$ satisfying $0 \leq \hat{g} \leq \hat{f}$ in L'' there exists a net $0 \leq a_\alpha \leq E$ in A'' such that $a_\alpha \cdot \hat{f} \rightarrow \hat{g}$ in $\sigma(L'', L')$.*

Theorem 6 [12] *If A is an f -algebra with unit then A'' is topologically full with respect to A'' .*

Proof: Suppose that $\hat{f}, \hat{g} \in A''$ with $0 \leq \hat{g} \leq \hat{f}$. Consider $\pi \in \text{Orth}(A'')$ with the map v then there is an $F \in A''$, $0 \leq F \leq E$ such that $v(F) = \pi$. In other words, $v(F)(\hat{f}) = F \cdot \hat{f} = \hat{g}$.

Since A is $|\sigma|(A'', A')$ dense in A'' [2],

there exists a net $\{a_\alpha\}$ in A , $0 \leq a_\alpha \leq e$ with $a_\alpha \rightarrow F$.

So we have $a_\alpha \cdot \hat{f} \rightarrow F \cdot \hat{f}$ and then $a_\alpha \cdot \hat{f} \rightarrow \hat{g}$ in $\sigma(A'', A')$.

Example 7 [12] *Let L be a Riesz space with separating order dual and L be a $Z(L)$ -module then the bilinear map, $Z(L)'' \times L'' \rightarrow L''$ $(\hat{\pi}, \hat{f}) \rightarrow (\hat{\pi} \cdot \hat{f})(f) = \hat{f}(\hat{\pi} \cdot f)$ shows that L'' is an f -module over $Z(L)''$. So, $\text{Orth}(L'') \subseteq L_b(L'', L''; Z(L'')) \subset L_b(L'')$.*

Theorem 8 [12] *Let L be an f -module over A . Let $f \in L'$ be arbitrary and consider the set $S(f) = \{\psi_{f,\hat{f}} : \hat{f} \in L''\}$. $S(f)$ is order ideal in A' .*

Definition 9 [12] *Let L be an f -module over A . For $f \in L'$, the closure of the set $S(f) = \{\psi_{f,\hat{f}} : \hat{f} \in L''\}$ in $\sigma(A', A'')$ topology is called the support of f and is denoted by $\text{supp}(f)$.*

Theorem 10 [12] *Let L be an f -module over A . For $f, g \in L'$ $\text{supp}(f) \cap \text{supp}(g) = \{0\}$ if and only if $f \perp g$.*

Proof: $\Rightarrow$: Suppose $\text{supp}(f) \cap \text{supp}(g) = \{0\}$. $\text{supp}(f)$ and $\text{supp}(g)$ are projection bands because of Dedekind completeness of A' .

Thus $\text{supp}(f) \perp \text{supp}(g)$. Let $\hat{f} \in L''$, then $\psi_{f,\hat{f}} \perp \psi_{g,\hat{f}} \Rightarrow |\psi_{f,\hat{f}}| \wedge |\psi_{g,\hat{f}}| = 0$. It follows that $\psi_{|f| \wedge |g|, \hat{f}} = 0$ and $|f| \wedge |g| = 0$.

$\Leftarrow$: Let $f \perp g$. The map $(\hat{f}, f) \rightarrow \psi_{f,\hat{f}}$ of $L'' \times L' \rightarrow A'$ is bipositive so,

$$|\psi_{f,\hat{f}}| \leq \psi_{|f|,|\hat{f}|} \leq \psi_{|f|,|\hat{f}| \vee |\hat{g}|} \text{ and}$$

$$|\psi_{g,\hat{g}}| \leq \psi_{|g|,|\hat{g}|} \leq \psi_{|g|,|\hat{f}| \vee |\hat{g}|} \text{ for each } \hat{f}, \hat{g} \in L''.$$

$$0 = |\psi_{f,\hat{f}}| \wedge |\psi_{g,\hat{g}}| \leq \psi_{|f|,|\hat{f}| \vee |\hat{g}|} \wedge \psi_{|g|,|\hat{f}| \vee |\hat{g}|} = \psi_{|f| \wedge |g|, |\hat{f}| \vee |\hat{g}|} = 0.$$

Therefore $\psi_{f,\hat{f}} \perp \psi_{g,\hat{g}}$ for each $\hat{f}, \hat{g} \in L''$. Then $S(f) \perp S(g)$ and $\text{supp}(f) \cap \text{supp}(g) = \{0\}$.

Definition 11 [12] *Let L and M be two f -modules over A and $T \in L_b(L, M)$. T is called an f -orthomorphism if $S(Tx) \subset S(x)$ for each x in L . The collection of all f -orthomorphisms will be denoted by $\text{Orth}(L, M; A)$.*

Let L and M be two f -modules over A . If $T \in \text{Orth}(L, M; A)$ then $T' \in \text{Orth}(M', L'; A'')$ is well known from [12] and so we have the following theorem.

Theorem 12 *Let L and M be two f -modules over A . Assume both are topologically full with respect to A .*

i) *If $T \in \text{Orth}(L, M; A)$ then $T' \in \text{Orth}(M', L'; A'')$.*

ii) *if $T' \in \text{Orth}(M', L'; A'')$ then $T'' \in \text{Orth}(L'', M''; A'')$.*

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