Extended Results on f-Orthomorphisms over the Second Order Dual of an f-algebra

Ömer Gök
Department of Mathematics, Yıldız Technical University
Esenler, Davutpaşa, Istanbul, Turkey
gok@yildiz.edu.tr

Şebnem Yıldız Pestil
Department of Mathematics, Yıldız Technical University
Esenler, Davutpaşa, Istanbul, Turkey
spestil@yildiz.edu.tr

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Abstract

Let \( A \) be an Archimedean f-algebra with unit and \( L, M \) be two f-modules over \( A \). In this paper we show that \( L'' \) is an f-module over \( A'' \) and \( A'' \) is topologically full over \( A'' \). Assume that \( T \) is an f-orthomorphism from \( L \) to \( M \) over \( A \). Also, we show that the second adjoint \( T'' \) of \( T \) is an f-orthomorphism from \( L'' \) to \( M'' \) over \( A'' \).[12].

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1 Introduction

Let \( L \) and \( M \) be two Archimedean Riesz spaces and \( A \) be an f-algebra. The positive cone of \( L \) will be denoted by \( L_+ \). A linear operator between two Riesz spaces is said to be order bounded if it maps order bounded subsets of \( L \) to order bounded subsets of \( M \). The vector spaces of all order bounded operators
from the Riesz space $L$ into Riesz space $M$ will be denoted by $L_b(L, M)$. An operator $T : L \to L$ on a Riesz space $L$ is said to be band preserving whenever $T$ leaves all bands of $L$ invariant, i.e., whenever $T(B) \subseteq B$ holds for each band $B$ of $L$. Recall that in a Riesz space, two elements $x$ and $y$ are said to be disjoint (in symbols $x \perp y$) whenever $|x| \land |y| = 0$ holds. An operator $T : L \to M$ between two Riesz spaces is said to preserve disjointness whenever $x \perp y$ in $L$ implies $Tx \perp Ty$ in $M$. $\pi \in L_b(L)$ is called an orthomorphism of $L$ if $x \perp y$ in $L$ imply that $\pi(x) \perp y$. $\text{Orth}(L)$ is a space of all order bounded and band preserving operators on $L$ and the set of orthomorphisms on $L$ is $\text{Orth}(L) = \{ T \in L_b(L) : x \perp y \Rightarrow Tx \perp Ty \}$. As $I$ is the identity operator in $L$, the principle order ideal generated by the identity operator $I$ in $\text{Orth}(L)$ is called the ideal center of $L$ and is denoted by $Z(L),[1,3]$. If $L$ is a Dedekind complete Riesz space, $\text{Orth}(L) = B(I)$ as $B(I)$ is the band generated by $I,[13]$. The vector space $L'$ of all order bounded linear functionals on $L$ is called the order dual of $L$, i.e., $L' = L_b(L, R)$. $L'$ is a Riesz space,[2,14]. The order bidual $L''$ of $L$ is the order dual of $L'$ that is $L'' = (L')'$ and the order continuous part of the order bidual of $L$ is denoted by $(L')''_o$. Recall that the expression $L'$ separates the points of $L$ means that for each $0 < x \neq 0$ there exists some $0 < f \in L'$ with $f(x) \neq 0$. The Riesz space $A$ is said to be Riesz algebra(lattice ordered algebra) if there exists an associative multiplication in $A$ with the usual algebra properties such that $ab \in A_+$ for all $a, b \in A_+$. Such a Riesz algebra $A$ is called an f-algebra if $A$ has the additional property that $a \perp b$ in $A$ implies $ca \perp b$ and $ac \perp b$ for all $c \in A$ (in other words, left and right multiplication by $c$ are orthomorphisms of $A$). Since every Archimedean f-algebra is commutative [7],we deal only with commutative f-algebras in this work. Let $A$ be an f-algebra with $A'$ separates the points of $A$. Let us recall also that $(A')'_n \subseteq \text{Orth}(A)$, the order continuous part of the order bidual of an f-algebra $A,[8]$. If $A$ has a multiplicative unit, then $(A')'_n = (A')'$, the whole order bidual of $A,[8]$. For unexplained notions, we refer to the books [2,9,10,11,14]. Here all Riesz spaces have separating order duals.

**Definition 1** [12,6] Let $A$ be an f-algebra with unit $e$ and $L$ be a Riesz space. $L$ is said to be an f-module over $A$ if there exists a map; $A \times L \to L : (a, x) \to a \cdot x$ satisfying,
1) $L$ is a module over $A$ and $e \cdot x = x$ for each $x \in L$
2) for each $a \in A_+$ and $x \in L_+$ we have $a \cdot x \in L_+$
3) If $x \perp y$ in $L$, then for each $a \in A$ we have $a \cdot x \perp y$.

**Lemma 2** [12] Let $A$ be an f-algebra with unit $e$ and $L$ be a Riesz space. For all $x \in L$, $f \in L'$ and $a \in A$. Riesz space $L'$ with $A \times L' \to L' : (a, f) \to a \cdot f : a \cdot f(x) = f(a \cdot x)$ is an f-module over $A$. 
Extended results on $f$-orthomorphisms

In this case, we define a map $m : A \to Orth(L')$ defined by $m(a) = \pi'_{\alpha}$ where $\pi'_\alpha(f) = a \cdot f$ for each $f \in L'$. $m$ is a unital algebra and Riesz homomorphism. This map induces an f-module structure on $L'$ over A. If $L$ is an f-module over $A$, we can define the mappings $[4]$.

1) $L \times L' \to A'$
\[ (x, f) \to x \cdot f : (xf)(a) = f(ax) \text{ for } x \in L, f \in L', a \in A. \]

2) $A'' \times L' \to L''$
\[ (F, f) \to F \cdot f : (F \cdot f)(x) = F(f \cdot x) \text{ for } x \in L, f \in L', F \in A''. \]

3) $A'' \times L'' \to L''$
\[ (F, \hat{f}) \to (F \cdot \hat{f})(f) = \hat{f}(F \cdot f) \text{ for } f \in L', F \in A'', \hat{f} \in L''. \]

We introduce the mappings:
1) $A \times L' \to L'$
\[ (a, f) \to a \cdot f : af(x) = f(ax) \text{ for } x \in L, f \in L', a \in A. \]

2) $L' \times L' \to A$
\[ (\hat{f}, f) \to \psi_{f, j} : \psi_{f, j}(a) = \hat{f}(a \cdot f) \text{ for } a \in A, f \in L', \hat{f} \in L''. \]

**Theorem 3** [12] Suppose that $L$ is an $f$-module over $A$.
Let $u : A \to Orth(L'')$ where $u(a)\hat{f} = a \cdot \hat{f}$ for each $\hat{f} \in L''$. $v : A'' \to Orth(L'')$ where $v_{\hat{f}}(f) = F \cdot \hat{f}$ for each $\hat{f} \in L'$. Then $u$ and $v$ are positive operators with $u(A) \subseteq Orth(L'')$, $v(A'') \subseteq Orth(L'')$.

Also, $u$ and $v$ are unital algebra and Riesz homomorphism.

**Proof:** Here $u$ and $v$ are positive operators.
Let $\pi'_\alpha \in Orth(L)$ and we denote the adjoint of $\pi'_\alpha$ by $\pi''_{\alpha}$.

$m : A \to Orth(L') : a \to m(a) = \pi'_\alpha$ where $\pi'_\alpha(f) = a \cdot f$ for $f \in L'$ and

$k : Orth(L') \to Orth(L'') : \pi' \to k(\pi') = \pi''$ are Riesz and algebra homomorphisms. $m$ is a unital algebra and Riesz homomorphism by Proposition 2.2[12].

The mapping $k$ is injective, positive and satisfies $k(I') = I''$ where $I'$ denotes the identity mapping of $L'$. $Orth(L'')$ is commutative so,

\[ k(\pi'_1 \pi'_2) = \pi''_{\pi'_1 \pi'_2} = \pi'' \pi''_{\pi'_1} = \pi''_{\pi'_1} \pi'' = k(\pi'_1)k(\pi'_2) \text{ for all } \pi'_1, \pi'_2 \in Orth(L'). \]

By Corollary 5.5 of [7] $k$ is an Riesz homomorphism.

For $a \in A$, $\hat{f} \in L''$, and $f \in L$ we have $u(a)\hat{f}(f) = a \cdot \hat{f}(f) = \hat{f}(af) = \hat{f}(\pi'_\alpha(f)) = \pi''_{\alpha}(\hat{f})(f) = (k \circ m)(\pi'_\alpha)(\hat{f})(f)$. $k$ and $m$ are algebra homomorphisms since $u(a \cdot b) = (k \circ m)(a \cdot b) = k(m(a \cdot b)) = k(m(a)) \cdot k(m(b)) = (k \circ m)(a)(k \circ m)(b) = u(a) \cdot u(b)$. Thus, $u$ is an algebra homomorphism. Let $\hat{a}$ be the image of $A''$ and for $a \in A$, $\hat{f} \in L''$, $f \in L'$,

\[ v_{\hat{a}}(f) = \hat{a} \cdot \hat{f}(f) = \hat{a}(\psi_{f, j})(a) = \hat{f}(af) = a \cdot \hat{f}(f) = u(a)\hat{f}(f), \]

and $v_{\hat{a}}(\hat{f}) \in Orth(L'')$.

Let $I(A)$ denote the ideal generated by $A$ in $A''$ and $0 \leq F \in I(A)$. Hence there exists $a \in A$ with $0 \leq F \leq \hat{a}$. As $v : A'' \to Orth(L'')$ is positive we have $0 \leq v_F \leq v_{\hat{a}}$ then $v_F \in Orth(L'')$. We choose $G_{\alpha}$ in $I(A)$ with $G_{\alpha} \uparrow F$. As $0 \leq \psi_{f, j} \in A'$ for each $f \in L'$, and $\hat{f} \in L''$, we have

$G_{\alpha}(\psi_{f, j}) \uparrow F(\psi_{f, j})$ so $G_{\alpha} \circ \hat{f} \uparrow F \circ \hat{f}$. 

\[ G_a \circ \hat{f} \uparrow F \circ \hat{f} \text{ and } v_{G_a} \uparrow v_F. \]

Since \( v_{G_a} \in \text{Orth}(L') \) and Orth\((L'') \) is a band we get \( v_F \in \text{Orth}(L'') \).

Consider the mapping \( v : A'' \rightarrow \text{Orth}(L') \) defined by \( v(F)(\hat{f}) = F \cdot \hat{f} = v_F(\hat{f}) \) for all \( F \in A'' \) and \( \hat{f} \in L'' \).

\[ ((F \cdot G) \cdot \hat{f})(\hat{f}) = F(G \cdot (\hat{f} \cdot \hat{f})) = F(G \cdot \hat{f}) = (F \cdot (G \cdot \hat{f}))(\hat{f}) \]

for all \( F, G \in A'' \) and \( \hat{f} \in L'' \).

Therefore, \( v_{F-G} = v_F \cdot v_G \) for all \( F, G \in A'' \). \( v \) is an algebra homomorphism.

Suppose that \( F \wedge G = 0 \) in \( A'' \). Then \( v(F \cdot G) = v(F)v(G) = 0 \) in Orth\((L'') \) as \( FG = 0 \). However, Orth\((L'') \) is a unital and hence semiprime f-algebra, \([5, 8]\) so \( v(F) \wedge v(G) = 0 \). We may conclude that \( v \) is disjoint preserving and from \([7]\) it is also Riesz homomorphism.

**Theorem 4** \([12]\) Let \( A \) be an f-algebra with unit and \( L \) be an f-module over \( A \). Then Riesz space \( L'' \) is an f-module over \( A'' \).

**Proof:** From the consequence of previous theorem, \( L'' \) is an f-module over \( A'' \).

That is,

\[ A'' \times L'' \rightarrow L'' : (F, \hat{f}) \rightarrow (F \cdot \hat{f})(\hat{f}) = \hat{f}(F \cdot f) \]

satisfies the following conditions;

1) \( F \cdot (G \cdot \hat{f}) = (F \cdot G)\hat{f} \), for all \( F, G \in A'' \) and \( \hat{f} \in L'' \)

2) \( 0 \leq F \in A'', 0 \leq \hat{f} \in L'' \), \( F \cdot \hat{f} \geq 0 \)

3) If \( \hat{f} \perp \hat{g} \) in \( L'' \), then \( F \cdot \hat{f} \perp \hat{g} \) in \( L'' \), for all \( \hat{f}, \hat{g} \in L'' \) and \( F \in A'' \).

The above theorem shows that, when \( A'' \) is an f-algebra with unit, \( L'' \) is a unital f-module over \( A'' \) and multiplication satisfies \( E \cdot \hat{f} = \hat{f} \) for all \( \hat{f} \in L'' \) and \( E \in A'' \)(\( E \) is the unit element in \( A'' \)).

**Definition 5** \([12]\) The f-module \( L'' \) over \( A'' \) with unit \( E \) is said to be topologically full with respect to \( A'' \) if for two arbitrary vectors \( \hat{f}, \hat{g} \) satisfying \( 0 \leq \hat{g} \leq \hat{f} \) in \( L'' \) there exists a net \( 0 \leq a_\alpha \leq E \) in \( A'' \) such that \( a_\alpha \cdot \hat{f} \rightarrow \hat{g} \) in \( \sigma(L'', L') \).

**Theorem 6** \([12]\) If \( A \) is an f-algebra with unit then \( A'' \) is topologically full with respect to \( A'' \).

**Proof:** Suppose that \( \hat{f}, \hat{g} \in A'' \) with \( 0 \leq \hat{g} \leq \hat{f} \). Consider \( \pi \in \text{Orth}(A'') \) with the map \( v \) then there is an \( F \in A'' \), \( 0 \leq F \leq E \) such that \( v(F) = \pi \). In other words, \( v(F)(\hat{f}) = F \cdot \hat{f} = \hat{g} \).

Since \( A \) is \( |\sigma|(A'', A') \) dense in \( A'' \) \([2]\),

there exists a net \( \{a_\alpha\} \) in \( A \), \( 0 \leq a_\alpha \leq e \) with \( a_\alpha \rightarrow F \).

So we have \( a_\alpha \cdot \hat{f} \rightarrow F \cdot \hat{f} \) and then \( a_\alpha \cdot \hat{f} \rightarrow \hat{g} \) in \( \sigma(A'', A') \).

**Example 7** \([12]\) Let \( L \) be a Riesz space with separating order dual and \( L \) be a \( Z(L) \)-module then the bilinear map, \( Z(L)'' \times L'' \rightarrow L'' : (\hat{\pi}, \hat{f}) \rightarrow (\hat{\pi} \cdot \hat{f})(\hat{f}) = \hat{f}(\hat{\pi} \cdot f) \) shows that \( L'' \) is an f-module over \( Z(L)' \). So, \( \text{Orth}(L'') \subseteq L^b(L'', L''; Z(L'')) \subset L^b(L'') \).
**Theorem 8** [12] Let $L$ be an $f$-module over $A$. Let $f \in L'$ be arbitrary and consider the set $S(f) = \{ \psi_{f,\hat{f}} : \hat{f} \in L'' \}$. $S(f)$ is order ideal in $A'$.

**Definition 9** [12] Let $L$ be an $f$-module over $A$. For $f \in L'$, the closure of the set $S(f) = \{ \psi_{f,\hat{f}} : \hat{f} \in L'' \}$ in $\sigma(A',A'')$ topology is called the support of $f$ and is denoted by $\text{supp}(f)$.

**Theorem 10** [12] Let $L$ be an $f$-module over $A$. For $f, g \in L'$, $\text{supp}(f) \cap \text{supp}(g) = \{0\}$ if and only if $f \perp g$.

**Proof:** $\Rightarrow$: Suppose $\text{supp}(f) \cap \text{supp}(g) = \{0\}$. $\text{supp}(f)$ and $\text{supp}(g)$ are projection bands because of Dedekind completeness of $A'$.

Thus $\text{supp}(f) \perp \text{supp}(g)$. Let $\hat{f} \in L''_+$, then $\psi_{f,\hat{f}} \perp \psi_{g,\hat{f}} \Rightarrow \vert \psi_{f,\hat{f}} \vert = 0$. It follows that $\psi_{f,\hat{f}} = 0$ and $|f| \bigwedge |g| = 0$.

$\Leftarrow$: Let $f \perp g$. The map $(\hat{f}, f) \to \psi_{f,\hat{f}}$ of $L'' \times L' \to A'$ is bijective so,

$\vert \psi_{f,\hat{f}} \vert \leq \psi_{|f|,|f|} \leq \psi_{|f|,|f| \bigvee |g|}$ and

$\vert \psi_{g,\hat{g}} \vert \leq \psi_{|g|,|g|} \leq \psi_{|g|,|f| \bigvee |g|}$ for each $\hat{f}, \hat{g} \in L''$. This implies

$0 = \psi_{f,\hat{f}} \bigwedge \psi_{g,\hat{g}} \leq \psi_{f,\hat{f} \bigvee |g|} \bigwedge \psi_{g,|g| \bigvee |\hat{g}|} = \psi_{f,|f| \bigvee |g|} = 0$.

Therefore $\psi_{f,\hat{f}} \perp \psi_{g,\hat{g}}$ for each $\hat{f}, \hat{g} \in L''$. Then $S(f) \perp S(g)$.

**Definition 11** [12] Let $L$ and $M$ be two $f$-modules over $A$ and $T \in L_0(L, M)$. $T$ is called an $f$-orthomorphism if $S(Tx) \subset S(x)$ for each $x$ in $L$. The collection of all $f$-orthomorphisms will be denoted by $\text{Orth}(L, M; A)$.

Let $L$ and $M$ be two $f$-modules over $A$. If $T \in \text{Orth}(L, M; A)$ then $T' \in \text{Orth}(M', L'; A'')$ is well known from [12] and so we have the following theorem.

**Theorem 12** Let $L$ and $M$ be two $f$-modules over $A$. Assume both are topologically full with respect to $A$.

i) If $T \in \text{Orth}(L, M; A)$ then $T' \in \text{Orth}(M', L'; A'')$.

ii) If $T' \in \text{Orth}(M', L'; A'')$ then $T'' \in \text{Orth}(L'', M''; A''')$.

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**References**


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