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Convergence Theorems and Base Properties on Condition (C)

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Abstract

A mapping T on subset C of Banach space E is called no expansive and quasi-no expansive, if for each $x, y, z \in F(T)$, we have $\|y - x\| \leq \|Ty - Tx\|$, $\|z - x\| \geq \|z - Tx\|$, where $F(T)$ is the set of fixed points of mapping T. That in this research we introduce some conditions on mappings that are weaker than no expansive condition and stronger than qusi-no expansive condition and by fixed points theorems and convergence theorems, we prove fixed point for these set of mappings.

Keywords: non-linear operators; Banach space E; no expansive Banach space; quasi-no expansive Banach space; increasing operators

1 Introduction

Metric fixed point theory is related to the results of fixed point theory in geometry condition on fundamental spaces and mapping in these spaces. The most famous theorem in fixed points theory is Banach contraction mapping principle and any discussion on contractive mappings is unavoidably converged to other no expansive mappings.

2 Main results

Definition 2.1 *Let T be a mapping on subset C of Banach space E, we say statisfies in property C, if for each $x, y \in C$ we have*

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow |Tx - Tx|| \leq ||x - y||.$$

Definition 2.2 *let T be a mapping on subset C of Banach space E, then T is called quasi-no expansive mapping if*

$$||Tx - z|| \leq ||x - z|| \quad for\ each\ x \in C, z \in F(T)$$

3 Convergence theorems and base properties on condition (C)

Definition 3.1 *Let E be a Banach space, we say E has Opial property, if for each weak convergent sequence $\{x_n\}$ on E with limit z and each $y \in E$ that $y \neq z$ we have*

$$\lim_{n\to\infty} inf||x_n - z|| < \lim_{n\to\infty} inf||x_n - y||$$

Theorem 3.2 *If mapping T with fixed pint satisfied in condition (C), then T is a quasi-no expansive mapping.*

Proof. Let $x \in B$, $z \in F(T)$ are constant. According to definition of condition (C) we have:

$$||z - Tx|| \leq ||z - x||$$

$$\frac{1}{2}||z - Tz|| = 0 \leq ||x - y|||z - x||$$

Thus, T mapping is quasi-no expansive.

Theorem 3.3 *Suppose T is a mapping on weakly compact convex subset B of Banach space E with opial property satisfying in condition (C). If we define sequence $\{x_n\}$ for each $n \in N$ as the followings:*

$$x_{n+1} = \mu Tx_n + (1 - \mu)x_n \ \ and \ \ x_1 \in B$$

where $\mu \in [\frac{1}{2}, 1)$, then sequence $\{x_n\}$ is weakly convergent to a fixed point T.

Proof. subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in B$ exist such that $\{x_{n_j}\}$ is weakly convergent to z,

$$||x_{n+1} - z|| = ||\mu(Tx_n - z) + (1 + \mu)(x_n - z)||$$

$$\leq \mu||Tx_n - z|| + (1-\mu)||x_n - z||$$

$$\mu||Tx_n - z|| + (1-\mu)||x_n - z|| \leq \mu||x_n - z|| + (1-\mu)||x_n - z||$$

$$= ||x_n - z||$$

Thus,

$$||x_{n+1} - z|| \leq \mu||Tx_n - z|| + (1-\mu)||x_n - z||$$

$$\leq ||x_n - z||$$

Now by Reductio ad absurdum, we suppose that $\{x_n\}$ is not convergent to z, then subsequence $\{x_n\}$ of $\{x_n\}$ and $\omega \in B$ exist such that is weakly convergent to ω that $\omega \neq z$. We see that $T\omega = \omega$. According to opial property we have

$$\lim_{n\to\infty} ||x_n - z|| = \lim_{j\to\infty} ||x_{n_j} - z||$$

$$< \lim_{j\to\infty} ||x_{n_j} - \omega|| = \lim_{n\to\infty} ||x_n - \omega|| = \lim_{k\to\infty} ||x_{n_k} - \omega|| < \lim_{k\to\infty} ||x_{n_k} - z|| = \lim_{n\to\infty} ||x_n - z||$$

That is contradiction.

4 Existence theorems

Theorem 4.1 *Suppose T is a mapping on weakly compact convex subset B of Banach space UCED that satisfies in condition (C). Then T has a fixed point.*

Proof. Sequence $\{x_n\}$ in B for $n \in N$ is defined as $x_{n+1} = \frac{1}{2}Tx_n + \frac{1}{2}x_n$. Then we have

$$\lim_{n\to\infty} sup||Tx_n - x_n|| = 0$$

Now convex and continuous function f is defined as: $f : B \to [0, \infty)$

$$f(x) = \lim_{n\to\infty} sup||x_n - x|| \ \ for\ each\ x \in B$$

As B is weakly compact and f is weakly lower semi-continuous, thus $z \in B$ exists such that $f(z) = min\{f(x) : x \in b\}$ then we have

$$||x_n - Tz|| \leq 3||x_n - Tx_n|| + ||x_n - z||$$

$$\lim_{n\to\infty} sup||x_n - Tz||$$

$$\leq \lim_{n\to\infty} sup(3||x_n - Tx_n|| + ||x_n - z|| \leq 3 \lim_{n\to\infty} sup||x_n - Tx_n|| + \lim_{n\to\infty} sup||x_n - z||$$

$$= 0 + \lim_{n\to\infty} sup||x_n - z|| = \lim_{n\to\infty} sup||x_n - z||$$

Thus $f(Tz) \leq f(z)$. As $f(z)$ is minimum, thus $f(Tz) = f(z)$. As B is compact and $\{x_n\} \subseteq B$ then $\{x_n\}$ is bounded. Thus, f is strictly quasi-convex. if $z \neq Tz$, as $f(z)$ is minimum we have

$$f(z) \leq f(\frac{z+Tz}{2}) < max\{f(z), f(Tz)\} = f(z)$$

That is contradiction. Thus, $z = Tz$.

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