A Remark on Bounded Homomorphisms into $B(\mathcal{H})$
and Amenability of Unitary Groups

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Abstract. Let $\mathcal{A}$ be a unital amenable $A^*$-algebra, or a unital commutative hermitian Banach $*$-algebra. We show that if $\rho : \mathcal{A} \to B(\mathcal{H})$ is a bounded homomorphism, then $\rho$ is completely bounded and $\|\rho\|_{cb} \leq \|\rho\|^2$.

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1. Introduction and Preliminaries

For a topological group $G$, $LUC(G)$ is the space of left uniformly continuous bounded functions $f : G \to C$. It is well known and easy to show that if $f : G \to C$ is bounded, then $f \in LUC(G)$ if and only if the map $x \to x_f$ is norm continuous from $G$ to $\ell^{\infty}(G)$, where $x_f(y) = f(yx)(y \in G)$. The space $LUC(G)$ is a unital $C^*$-subalgebra of $\ell^{\infty}(G)$, and is right invariant in the sense
that $xf \in RUC(G)$ whenever $f \in RUC(G)$. If $X$ is a right invariant, unital subspace of $\ell^\infty(G)$, then an element $m \in X^*$ is called a right invariant mean if $m(1) = 1 = \|m\|$ and $m(xf) = m(f)$ for all $f \in X$, and $x \in G$. We denote by $R(X)$ the set of right invariant means on $X$.

Supposed that $A$ is a $C^*$-algebra. We denote by $M_n(A)$ the $C^*$-algebra of $n \times n$ matrices with entries in $A$. For $C^*$-algebra $A$ and $B$, a (linear) map $\Psi : A \to B$ is called: positive if $\Psi(a) \geq 0$ for every $a \geq 0$; completely positive if $\Psi(n) : M_n(A) \to M_n(B)$ is positive for every $n \in \mathbb{N}$, where $\Psi(n)([a_{ij}]) = [\Psi(a_{ij})]$; completely bounded if $\|\Psi\|_{cb} = \sup_n \|\Psi(n)\| < \infty$.

We start by the following Theorem [10, Theorem 9.7]:

**Theorem 1.** Let $A$ be a commutative unital $C^*$-algebra, and if $\rho : A \to B(\mathcal{H})$ is a bounded homomorphism, then $\rho$ is completely bounded and $\|\rho\|_{cb} \leq \|\rho\|^2$.

In the proof this Theorem actually used from amenability of the unitary groups of $C^*$-algebra $A$. The relationship between amenability of Banach algebras and amenability of groups (semigroups) is well known and consider. Johnson [6, Theorem 2.5] proved the remarkable result that if $G$ is a locally compact group then $G$ is amenable if and only if $L^1(G)$ is amenable Banach algebra. Paterson [9, Theorem 2] shows that a $C^*$-algebra $A$ is amenable if and only if $U(A)$, the unitary group of $A$, is amenable. In the other hand we have the following example.

**Example 1.** Let $A$ be the disc algebra, we define an involution $f \to f^*$ on $A$ by $f^*(z) = \overline{f(\overline{z})}$, then $A$ is a Banach $*$-algebra. It is well known that $A$ is not amenable. Now if $S$ is the isometry semigroup of $A$, then $S$ is a semitopological semigroup w.r.t. the relative (Banach space) weak topology since $S$ is abelian thus is amenable, i.e. $R(LUC(S)) \neq \emptyset$.

We have attempted to develop Theorem 1, for unital amenable $A^*$-algebras, and for unital commutative hermitian Banach $*$-algebras.

The importance of the amenability (or the existence of an invariant mean) is best illustrated in the following Theorem.

**Theorem 2.** (Dixmier) Let $G$ be an amenable locally compact group, and let $\rho : G \to B(\mathcal{H})$ be a strongly continuous homomorphism with $\rho(e) = 1$, such that $\|\rho\|_u := \sup\{\|\rho(t)\| : t \in G\}$ is finite. Then there exists an invertible $S$ in $B(\mathcal{H})$ with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|_u^2$, such that $S^{-1}\rho(\cdot)S$ is a unitary representation of $G$.

The proof of this theorem [10, Theorem 9.3] holds also for the following version:

**Theorem 3.** Let $G$ be an amenable topological group, i.e., $R(LUC(G)) \neq \emptyset$ and let $\rho : G \to B(\mathcal{H})$ be a strongly continuous homomorphism with $\rho(e) = 1$, such that $\|\rho\|_u := \sup\{\|\rho(t)\| : t \in G\}$ is finite. Then there exists an invertible $S$ in $B(\mathcal{H})$ with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|_u^2$, such that $S^{-1}\rho(\cdot)S$ is a unitary representation of $G$. 
A remark on bounded homomorphisms into $B(H)$

2. Main results

First of all we state the following proposition.

**Proposition 1.** Let $A$ be a unital Banach $*$-algebra such that the unitary group $U(A)$ is a bounded set. Then $A$ is amenable if and only if $U(A)$ is amenable, i.e. $R(LUC(U(A))) \neq \emptyset$.

**Proof.** Since $U(A)$ is a bounded set, by [4, Theorem 34.3], $A$ admits an equivalent $C^*$-norm $\|\|_*$. So amenability of $(A, \|\|_*)$ is equivalent to amenability of $(A, \|\|_*)$ and by [9, Theorem 2] is equivalent to amenability of $U(A)$. \[\square\]

**Example 2.** Let $J$ be an arbitrary set of elements and consider the space of those complex-valued functions $a(i, j)$ defined on $J \times J$ which satisfy the condition $\sum_{i,j} |a(i, j)|^2 \leq \infty$. We make this set into an $H^*$-algebra by the following definitions: if $a = a(i, j), b = b(i, j)$, and $\lambda$ is any complex number then

\[
(a + b)(i, j) = a(i, j) + b(i, j),
\]

\[
(ab)(i, j) = \sum_k a(i, k)b(k, j),
\]

\[
(\lambda a)(i, j) = \lambda a(i, j),
\]

\[
\langle a, b \rangle = \alpha \sum_{i,j} a(i, j)b(i, j) \quad (\alpha \geq 1),
\]

\[
a^*(i, j) = \overline{a(j, i)}.
\]

It is easy to verify that with these definitions this set becomes an $H^*$-algebra. If $n$ is the cardinal number of $J$ then this algebra is called the full matrix $H^*$-algebra of order $n$, or sometimes simply a full matrix algebra. This algebra is amenable since its unitary group is a bounded set. Therefore every simple $H^*$-algebra is amenable.

We recall that a Banach $*$-algebra $(A, \|\|)$ is said to be an $A^*$-algebra provided there exists on $A$ a second norm $\|\|_*$ not necessarily complete, which is a $C^*$-norm. The second norm will be called an auxiliary norm. The completion of $(A, \|\|_*)$ is a $C^*$-algebra and is denoted by $C^*(A)$. Any $C^*$-algebra is an $A^*$-algebra. An important example of a Banach $*$-algebra whose norm does not satisfy the condition $\|x\|^2 = \|x^*x\|$, but is an $A^*$-algebra, is the group algebra $L^1(G)$ of a locally compact group $G$. Another example of an $A^*$-algebra is provided by any semisimple real commutative Banach algebra. In this case, the involution is the identity map and auxiliary norm is the spectral radius $r$. Also, a semisimple hermitian Banach $*$-algebra is an $A^*$-algebra, [3, Section 41, Corollary 10]. Gelfand and Naimark [5] give an example of an $A^*$-algebra, which is not hermitian. We denote by $A_{sa}$ the self-adjoint part of $A$.

If $B$ is a unital $C^*$-algebra with unitary group $U(B)$, we consider on $U(B)$ the relative weak topology (as a subset of the Banach space $B$). Regarding $B \subset$
$B^{**}$, $B^{**}$ is a von Neumann algebra. Give on the unitary group $U(B^{**})$, the relative ultraweak topology ($\sigma(B^{**}, B^*)$)-topology. Now on $U(B^{**})$ this topology coincides with both the weak operator and the strong operator topology on $U(B^{**})$. Since involution is weak operator continuous and multiplication is strong operator continuous on $U(B^{**})$, it follows that $U(B^{**})$ is a topological group. Further since the weak topology on $B$ coincides with the relative ultra-
weak topology, it follows that the topology on $U(B)$ is the relative topology inherited from $U(B^{**})$. Hence, $U(B)$ is a topological group.

**Proposition 2.** Let $A$ be a unital $A^*$-algebra with the auxiliary norm $\|\cdot\|_*$, $G = U(A)$ the unitary group of $A$ and $B = C^*(A)$. If we give $G$, the relative weak topology (as a subset of the Banach space $B$), then $G$ is a topological group and is dense in $U(B)$ w.r.t. this topology.

**Proof.** $G$ as a subgroup of topological group $U(B)$ is a topological group [2, Proposition 1.3.4]. Suppose that $x \in U'(B)$ where $U'(B) = \{u \in U(B) : \|1 - u\|_* < 2\}$ then there exists a $y \in B_{sa}$ such that $x = e^{iy}$ [7, Theorem 2.1.12]. Since $A_{sa}$ is $\|\cdot\|_*$-dense in $B_{sa}$ therefore there exists $(y_n) \subset A_{sa}$ such that $y_n \to y$ in $\|\cdot\|_*$-topology, so $(y_n)$ is $\|\cdot\|_*$-bounded (say by $M$), thus $\sigma_B(y_n) \subset [-M, M]$.

Define $f : [-M, M] \to C$ by $f(t) = e^{it}$, we know that $f \in C([-M, M])$ hence $f(y_n) \to f(y)$ or $e^{iy_n} \to e^{iy} = x$, in $\|\cdot\|_*$-topology. Note that there are two exponential functions here for $e^{iy}$, one for each norm, but since $\|\cdot\|_*$ is $\|\cdot\|_*$-continuous they are agree. As $y_n \in A_{sa}$ thus $e^{iy_n}$ is unitary element of $A$. Consequently $\|\cdot\|_*$-closure of $U(A)$ contains $U'(B)$. By [11, Theorem 4.11] $U'(B)$ is dense in $U(B)$, w.r.t. the relative (Banach space) weak topology, so $U(B) = w$-closure of $U(A)$. \[ \square \]

**Corollary 1.** Let $A$ be a unital *-subalgebra of a unital $C^*$-algebra B. then $G = U(A)$ the unitary group of $A$ is a topological group and is dense in $U(B)$ w.r.t. the relative weak topology.

The following Corollary is a generalization of [9, Theorem 2].

**Corollary 2.** Let $A$ be a unital $A^*$-algebra. If $A$ is amenable then $G = U(A)$ the unitary group of $A$ is amenable i.e. $\mathcal{R}(LUC(G)) \neq \emptyset$.

**Proof.** If $A$ is amenable since $A$ is a dense subalgebra of $C^*(A)$ thus $C^*(A)$ is amenable. By [9, Theorem 2], $U(C^*(A))$ is amenable, it follows from [9, Proposition 1] that if $H$ is a topological group and $G$ is a subgroup of $H$ such that $\overline{G} = H$, then

$$\mathcal{R}(LUC(G)) \neq \emptyset$$

if and only if $\mathcal{R}(LUC(H)) \neq \emptyset$.

Since $U(C^*(A))$ is amenable and by previous Proposition 2, $U(C^*(A)) = w$-closure of $U(A)$, hence $U(A)$ is amenable. \[ \square \]

**Example 3.** Let $A = L^1(H) \oplus C1$ where $L^1(H)$ is the set of all trace class operators on a separable Hilbert space $H$, with the trace norm $\|\cdot\|_tr$, and auxiliary norm given by the operator norm. Then, $A$ is a $A^*$-algebra and
Proof. Since $\|u\|$ is unitary in the closed unit ball of $B = C^*(A)$, therefore $\|\|\|$-bounded sets in $A$ are equals. Hence, $\rho : A \to B(H)$ is a bounded homomorphism. We may $\rho$ extend to a bounded linear map $\tilde{\rho} : C^*(A) \to B(H)$. Clearly $\tilde{\rho}$ is a homomorphism, and by [10, Theorem 9.7] $\tilde{\rho}$ is completely bounded and $\|\tilde{\rho}\|_{cb} \leq \|\tilde{\rho}\|^2$. By Proposition 2 $U(C^*(A)) = w$-closure of $U(A)$, and since $\tilde{\rho}$ is w-continuous, therefore $\|\tilde{\rho}\| := \sup\{\|\tilde{\rho}(t)\| : t \in U(A)\} = \sup\{\|\tilde{\rho}(t)\| : t \in U(C^*(A))\} = \|\tilde{\rho}\|_u$.

We recall that the involution in a $*$-algebra $A$ is said to be hermitian if every hermitian element in $A$ has real spectrum. When the involution in a $*$-algebra $A$ is hermitian then $A$ is called hermitian algebra. Many $*$-algebras are hermitian, for instance all $C^*$-algebras are hermitian, as are the group algebras of abelian or compact groups. First we state the following Remark.

Remark 1. Let $A$ a unital hermitian Banach $*$-algebra, with an isometric involution. If $\rho : A \to B(H)$ is a $*$-representation, then the operator norm $\|\rho\| \leq 1$: Clearly $\rho$ maps unitaries to unitaries. Hence, $\|\rho(u)\| \leq 1$ for any unitary $u$. Let $h = h^*$ be a self-adjoint element of $A$, with $\|h\| \leq 1$, then $1 - h^2 \geq 0$. By [1, Theorem 6.1.4.] there exists a self adjoint element $\sqrt{1 - h^2} \in A$, therefore

$$u = h + i\sqrt{1 - h^2} \space (h = h^* \in A)$$

is easily seen to be unitary. This shows that every self-adjoint element in the unit ball is the real part of a unitary. We obtain that $\|\rho(h)\| \leq 1$ for every self-adjoint element $h$ in the unit ball $A$. Since $\|\rho(x)\|^2 = \|\rho(x)^*\rho(x)\| = \|\rho(x^*x)\| \leq \|x^*x\| \leq \|x\|^2 \leq \|x\|^2$, therefore the operator norm $\|\rho\| \leq 1$.

Thus we have the following proposition.

Proposition 4. Let $A$ be a unital commutative hermitian Banach $*$-algebra, with an isometric involution. If $\rho : A \to B(H)$ is a bounded homomorphism, then $\rho$ is completely bounded and $\|\rho\|_{cb} \leq \|\rho\|^2$.

Proof. By (2.1) above, we have $h = 1/2(u + u^*)$. This shows that every self-adjoint element is in the span of two unitaries. Using the Cartesian decomposition, we obtain that every element in $A$ is a linear combination of at most four unitaries. let $G$ denote the unitary group of $A$. Then, by (Dixmier’s Theorem) Theorem 2, there is a similarity $S$, with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|^2$, such
that \( \tilde{\rho}(t) := S^{-1}\rho(t)S \) is unitary for all \( t \in G \). Since \( \tilde{\rho}(t^*) = \tilde{\rho}(t)^{-1} = \tilde{\rho}(t)^* (t \in G) \), thus \( \tilde{\rho} \) is self-adjoint on the unitary elements of \( \mathcal{A} \). Since every element in \( \mathcal{A} \) is a linear combination of at most four unitaries, therefore \( \tilde{\rho} \) is a \( * \)-homomorphism on \( \mathcal{A} \).

In view of the obtained results, the following question is natural: Does the assertion of Corollary 2 holds for any unital hermitian Banach \( * \)-algebra?

**References**


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