A Remark on Bounded Homomorphisms into $B(\mathcal{H})$ and Amenability of Unitary Groups

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Abstract. Let $\mathcal{A}$ be a unital amenable A^* -algebra, or a unital commutative hermitian Banach $*$ -algebra. We show that if $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ is a bounded homomorphism, then ρ is completely bounded and $\|\rho\|_{cb} \leq \|\rho\|^2$.

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1. INTRODUCTION AND PRELIMINARIES

For a topological group G , $LUC(G)$ is the space of left uniformly continuous bounded functions $f : G \rightarrow C$. It is well known and easy to show that if $f : G \rightarrow C$ is bounded, then $f \in LUC(G)$ if and only if the map $x \rightarrow {}_x f$ is norm continuous from G to $\ell^\infty(G)$, where ${}_x f(y) = f(yx)$ ($y \in G$). The space $LUC(G)$ is a unital C^* -subalgebra of $\ell^\infty(G)$, and is right invariant in the sense

that ${}_xf \in RUC(G)$ whenever $f \in RUC(G)$. If X is a right invariant, unital subspace of $\ell^\infty(G)$, then an element $m \in X^*$ is called a right invariant mean if $m(1) = 1 = \|m\|$ and $m({}_xf) = m(f)$ for all $f \in X$, and $x \in G$. We denote by $\mathcal{R}(X)$ the set of right invariant means on X .

Supposed that $\mathcal{A}$ is a C^* -algebra. We denote by $M_n(\mathcal{A})$ the C^* -algebra of $n \times n$ matrices with entries in $\mathcal{A}$. For C^* -algebra $\mathcal{A}$ and $\mathcal{B}$, a (linear) map $\Psi : \mathcal{A} \rightarrow \mathcal{B}$ is called: positive if $\Psi(a) \geq 0$ for every $a \geq 0$; completely positive if $\Psi^{(n)} : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ is positive for every $n \in \mathbb{N}$, where $\Psi^{(n)}([a_{ij}]) = [\Psi(a_{ij})]$; completely bounded if $\|\Psi\|_{cb} = \sup_n \|\Psi^{(n)}\| < \infty$.

We start by the following Theorem [10, Theorem 9.7]:

Theorem 1. *Let $\mathcal{A}$ be a commutative unital C^* -algebra, and if $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ is a bounded homomorphism, then ρ is completely bounded and $\|\rho\|_{cb} \leq \|\rho\|^2$.*

In the proof this Theorem actually used from amenability of the unitary groups of C^* -algebra $\mathcal{A}$. The relationship between amenability of Banach algebras and amenability of groups (semigroups) is well known and consider. Johnson [6, Theorem 2.5] proved the remarkable result that if G is a locally compact group then G is amenable if and only if $L^1(G)$ is amenable Banach algebra. Paterson [9, Theorem 2] shows that a C^* -algebra A is amenable if and only if $U(A)$, the unitary group of A , is amenable. In the other hand we have the following example.

Example 1. Let A be the disc algebra, we define an involution $f \rightarrow f^*$ on A by $f^*(z) = \overline{f(\bar{z})}$, then A is a Banach $*$ -algebra. It is well known that A is not amenable. Now if S is the isometry semigroup of A , then S is a semitopological semigroup w.r.t. the relative (Banach space) weak topology since S is abelian thus is amenable, i.e. $\mathcal{R}(LUC(S)) \neq \emptyset$.

We have attempted to develop Theorem 1, for unital amenable $\mathcal{A}^*$ -algebras, and for unital commutative hermitian Banach $*$ -algebras. The importance of the amenability (or the existence of an invariant mean) is best illustrated in the following Theorem.

Theorem 2. (Dixmier) *Let G be an amenable locally compact group, and let $\rho : G \rightarrow B(\mathcal{H})$ be a strongly continuous homomorphism with $\rho(e) = 1$, such that $\|\rho\|_u := \sup\{\|\rho(t)\| : t \in G\}$ is finite. Then there exists an invertible S in $B(\mathcal{H})$ with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|_u^2$, such that $S^{-1}\rho(\cdot)S$ is a unitary representation of G .*

The proof of this theorem [10, Theorem 9.3] holds also for the following version:

Theorem 3. *Let G be an amenable topological group, i.e., $\mathcal{R}(LUC(G)) \neq \emptyset$ and let $\rho : G \rightarrow B(\mathcal{H})$ be a strongly continuous homomorphism with $\rho(e) = 1$, such that $\|\rho\|_u := \sup\{\|\rho(t)\| : t \in G\}$ is finite. Then there exists an invertible S in $B(\mathcal{H})$ with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|_u^2$, such that $S^{-1}\rho(\cdot)S$ is a unitary representation of G .*

2. MAIN RESULTS

First of all we state the following proposition.

Proposition 1. *Let A be a unital Banach $*$ -algebra such that the unitary group $U(A)$ is a bounded set. Then A is amenable if and only if $U(A)$ is amenable, i.e. $\mathcal{R}(LUC(U(A))) \neq \emptyset$.*

Proof. Since $U(A)$ is a bounded set, by [4, Theorem 34.3], A admits an equivalent C^* -norm $\|\cdot\|_*$. So amenability of $(A, \|\cdot\|)$ is equivalent to amenability of $(A, \|\cdot\|_*)$ and by [9, Theorem 2] is equivalent to amenability of $U(A)$. $\square$

Example 2. Let J be an arbitrary set of elements and consider the space of those complex-valued functions $a(i, j)$ defined on $J \times J$ which satisfy the condition $\sum_{i,j} |a(i, j)|^2 \leq \infty$. We make this set into an H^* -algebra by the following definitions: if $a = a(i, j), b = b(i, j)$, and λ is any complex number then

$$\begin{aligned} (a + b)(i, j) &= a(i, j) + b(i, j), \\ (ab)(i, j) &= \sum_k a(i, k)b(k, j), \\ (\lambda a)(i, j) &= \lambda a(i, j), \\ \langle a, b \rangle &= \alpha \sum_{i,j} a(i, j)b(i, j) \quad (\alpha \geq 1), \\ a^*(i, j) &= \bar{a}(j, i). \end{aligned}$$

It is easy to verify that with these definitions this set becomes an H^* -algebra. If n is the cardinal number of J then this algebra is called the full matrix H^* -algebra of order n , or sometimes simply a full matrix algebra. This algebra is amenable since its unitary group is a bounded set. Therefore every simple H^* -algebra is amenable.

We recall that a Banach $*$ -algebra $(\mathcal{A}, \|\cdot\|)$ is said to be an A^* -algebra provided there exists on $\mathcal{A}$ a second norm $\|\cdot\|_*$ not necessarily complete, which is a C^* -norm. The second norm will be called an auxiliary norm. The completion of $(\mathcal{A}, \|\cdot\|_*)$ is a C^* -algebra and is denoted by $C^*(\mathcal{A})$. Any C^* -algebra is an A^* -algebra. An important example of a Banach $*$ -algebra whose norm does not satisfy the condition $\|x\|^2 = \|x^*x\|$, but is an A^* -algebra, is the group algebra $L^1(G)$ of a locally compact group G . Another example of an A^* -algebra is provided by any semisimple real commutative Banach algebra. In this case, the involution is the identity map and auxiliary norm is the spectral radius r . Also, a semisimple hermitian Banach $*$ -algebra is an A^* -algebra, [3, Section 41, Corollary 10]. Gelfand and Naimark [5] give an example of an A^* -algebra, which is not hermitian. We denote by A_{sa} the self-adjoint part of A .

If B is a unital C^* -algebra with unitary group $U(B)$, we consider on $U(B)$ the relative weak topology (as a subset of the Banach space B). Regarding $B \subset$

B^{**} , B^{**} is a von Neumann algebra. Give on the unitary group $U(B^{**})$, the relative ultraweak topology ($\sigma(B^{**}, B^*)$ -topology). Now on $U(B^{**})$ this topology coincides with both the weak operator and the strong operator topology on $U(B^{**})$. Since involution is weak operator continuous and multiplication is strong operator continuous on $U(B^{**})$, it follows that $U(B^{**})$ is a topological group. Further since the weak topology on B coincides with the relative ultraweak topology, it follows that the topology on $U(B)$ is the relative topology inherited from $U(B^{**})$. Hence, $U(B)$ is a topological group.

Proposition 2. *Let A be a unital A^* -algebra with the auxiliary norm $\|\cdot\|_*$, $G = U(A)$ the unitary group of A and $B = C^*(A)$. If we give G , the relative weak topology (as a subset of the Banach space B), then G is a topological group and is dense in $U(B)$ w.r.t. this topology.*

Proof. G as a subgroup of topological group $U(B)$ is a topological group [2, Proposition 1.3.4]. Suppose that $x \in U'(B)$ where $U'(B) = \{u \in U(B) : \|1 - u\|_* < 2\}$ then there exists a $y \in B_{sa}$ such that $x = e^{iy}$ [7, Theorem 2.1.12]. Since A_{sa} is $\|\cdot\|_*$ -dense in B_{sa} therefore there exists $(y_n) \subset A_{sa}$ such that $y_n \rightarrow y$ in $\|\cdot\|_*$ -topology, so (y_n) is $\|\cdot\|_*$ -bounded (say by M), thus $\sigma_B(y_n) \subset [-M, M]$.

Define $f : [-M, M] \rightarrow C$ by $f(t) = e^{it}$, we know that $f \in C([-M, M])$ hence $f(y_n) \rightarrow f(y)$ or $e^{iy_n} \rightarrow e^{iy} = x$, in $\|\cdot\|_*$ -topology. Note that there are two exponential functions here for e^{iy} , one for each norm, but since $\|\cdot\|_*$ is $\|\cdot\|$ -continuous they are agree. As $y_n \in A_{sa}$ thus e^{iy_n} is unitary element of A . Consequently $\|\cdot\|_*$ -closure of $U(A)$ contains $U'(B)$. By [11, Theorem 4.11] $U'(B)$ is dense in $U(B)$, w.r.t. the relative (Banach space) weak topology, so $U(B) = \text{w-closure of } U(A)$. $\square$

Corollary 1. *Let A be a unital $*$ -subalgebra of a unital C^* -algebra B . then $G = U(A)$ the unitary group of A is a topological group and is dense in $U(B)$ w.r.t. the relative weak topology.*

The following Corollary is a generalization of [9, Theorem 2].

Corollary 2. *Let A be a unital A^* -algebra. If A is amenable then $G = U(A)$ the unitary group of A is amenable i.e. $\mathcal{R}(LUC(G)) \neq \emptyset$.*

Proof. If A is amenable since A is a dense subalgebra of $C^*(A)$ thus $C^*(A)$ is amenable. By [9, Theorem 2], $U(C^*(A))$ is amenable, it follows from [9, Proposition 1] that if H is a topological group and G is a subgroup of H such that $\overline{G} = H$, then

$$\mathcal{R}(LUC(G)) \neq \emptyset \text{ if and only if } \mathcal{R}(LUC(H)) \neq \emptyset.$$

Since $U(C^*(A))$ is amenable and by previous Proposition 2, $U(C^*(A)) = \text{w-closure of } U(A)$, hence $U(A)$ is amenable. $\square$

Example 3. Let $A = L^1(H) \oplus C1$ where $L^1(H)$ is the set of all trace class operators on a separable Hilbert space H , with the trace norm $\|\cdot\|_{tr}$, and auxiliary norm given by the operator norm. Then, A is a A^* -algebra and

$C^*(A) = K(H) \oplus C1$. Also, the $*$ -algebra $L^1(H)$ is hermitian since a $*$ -ideal (and hence spectral $*$ -subalgebra) of the C^* -algebra $B(H)$ [8, Section 10.5]. We know that A is not amenable because, it is well known that $L^1(H)$ does not admit a bounded approximate identity. But $C^*(A) = K(H) \oplus C1$ is amenable, therefore, $\mathcal{R}(LUC(U(A))) \neq \emptyset$.

Proposition 3. *Let $\mathcal{A}$ a unital amenable A^* -algebra. If $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ is a bounded homomorphism, then ρ is completely bounded and $\|\rho\|_{cb} \leq \|\rho\|_u^2$.*

Proof. Since $\mathcal{A}$ is a dense subalgebra of $C^*(\mathcal{A})$, thus closed unit ball of $\mathcal{A}$ is $\|\cdot\|_*$ -dense in closed unit ball of $\mathcal{B} = C^*(\mathcal{A})$, therefore $\|\cdot\|_*$ -bounded sets and $\|\cdot\|$ -bounded sets in $\mathcal{A}$ are equals. Consequently $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ is a $\|\cdot\|_*$ -bounded homomorphism. We may ρ extend to a bounded linear map $\bar{\rho} : C^*(\mathcal{A}) \rightarrow B(\mathcal{H})$. Clearly $\bar{\rho}$ is a homomorphism, and by [10, Theorem 9.7] $\bar{\rho}$ is completely bounded and $\|\bar{\rho}\|_{cb} \leq \|\bar{\rho}\|^2$. By Proposition 2 $U(C^*(\mathcal{A})) = \text{w-closure of } U(\mathcal{A})$, and since $\bar{\rho}$ is w-w-continuous, therefore $\|\rho\|_u := \sup\{\|\rho(t)\| : t \in U(\mathcal{A})\} = \sup\{\|\bar{\rho}(t)\| : t \in U(C^*(\mathcal{A}))\} = \|\bar{\rho}\|_u$. $\square$

We recall that the involution in a $*$ -algebra $\mathcal{A}$ is said to be hermitian if every hermitian element in $\mathcal{A}$ has real spectrum. When the involution in a $*$ -algebra $\mathcal{A}$ is hermitian then $\mathcal{A}$ is called hermitian algebra. Many $*$ -algebras are hermitian, for instance all C^* -algebras are hermitian, as are the group algebras of abelian or compact groups. First we state the following Remark.

Remark 1. Let $\mathcal{A}$ a unital hermitian Banach $*$ -algebra, with an isometric involution. If $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ is a $*$ -representation, then the operator norm $\|\rho\| \leq 1$: Clearly ρ maps unitaries to unitaries. Hence, $\|\rho(u)\| \leq 1$ for any unitary u . Let $h = h^*$ be a self-adjoint element of $\mathcal{A}$, with $\|h\| \leq 1$, then $1 - h^2 \geq 0$. By [1, Theorem 6.1.4.] there exists a self adjoint element $\sqrt{1 - h^2} \in \mathcal{A}$, therefore

$$(2.1) \quad u = h + i\sqrt{1 - h^2} \quad (h = h^* \in \mathcal{A})$$

is easily seen to be unitary. This shows that every self-adjoint element in the unit ball is the real part of a unitary. We obtain that $\|\rho(h)\| \leq 1$ for every self-adjoint element h in the unit ball $\mathcal{A}$. Since $\|\rho(x)\|^2 = \|\rho(x)^*\rho(x)\| = \|\rho(x^*x)\| \leq \|x^*x\| \leq \|x^*\|\|x\| \leq \|x\|^2$, therefore the operator norm $\|\rho\| \leq 1$.

Thus we have the following proposition.

Proposition 4. *Let $\mathcal{A}$ be a unital commutative hermitian Banach $*$ -algebra, with an isometric involution. If $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ is a bounded homomorphism, then ρ is completely bounded and $\|\rho\|_{cb} \leq \|\rho\|_u^2$.*

Proof. By (2.1) above, we have $h = 1/2(u + u^*)$. This shows that every self-adjoint element is in the span of two unitaries. Using the Cartesian decomposition, we obtain that every element in $\mathcal{A}$ is a linear combination of at most four unitaries. let G denote the unitary group of $\mathcal{A}$. Then, by (Dixmier's Theorem) Theorem 2, there is a similarity S , with $\|S\| \|S^{-1}\| \leq \|\rho\|^2$, such

that $\tilde{\rho}(t) := S^{-1}\rho(t)S$ is unitary for all t in G . Since $\tilde{\rho}(t^*) = \tilde{\rho}(t^{-1}) = \tilde{\rho}(t)^{-1} = \tilde{\rho}(t)^*$ ($t \in G$), thus $\tilde{\rho}$ is self-adjoint on the unitary elements of $\mathcal{A}$. Since every element in $\mathcal{A}$ is a linear combination of at most four unitaries, therefore $\tilde{\rho}$ is a $*$ -homomorphism on $\mathcal{A}$. $\square$

In view of the obtained results, the following question is natural: Does the assertion of Corollary 2 holds for any unital hermitian Banach $*$ -algebra?

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