Parallelism of Weil Bundles

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Abstract

Let $M$ be a smooth manifold, $A$ a Weil algebra and $M^A$ the associated Weil bundle. We use the structure of $C^\infty(M^A, A)$-module on the set $\mathfrak{X}(M^A)$ of vector fields on $M^A$ for to give the equivalence of parallelism of the $A$-manifold $M^A$.

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1 Preliminaries

A Weil algebra or a local algebra (in the sense of André Weil) is a unitary commutative real algebra of finite dimension with unique maximal ideal of codimension 1.

Let $A$ be a Weil algebra and let $\mathfrak{m}$ be its unique maximal ideal. We have

$$A = \mathbb{R} \oplus \mathfrak{m}.$$ 

The first projection

$$A = \mathbb{R} \oplus \mathfrak{m} \longrightarrow \mathbb{R}$$

is a surjective homomorphism of algebras, called augmentation and the unique integer $k \in \mathbb{N}$ such that $\mathfrak{m}^k \neq (0)$ and $\mathfrak{m}^{k+1} = (0)$ is the order of $A$.

We have, as examples of Weil algebras:
Example 1:

1. \( R = \mathbb{R} \oplus (0) \) is a Weil algebra of order 0.

2. The algebra of dual numbers, \( \mathbb{D} = \mathbb{R}[T]/(T^2) \), is a Weil algebra of order 1.

3. \( A = \mathbb{R}[T]/(T^3) \) is a Weil algebra of order 2. More generally, the algebra of truncated polynomials

\[
A = \mathbb{R}[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^{k+1}
\]

is a Weil algebra of order \( k \).

4. If \( M \) is a smooth manifold of dimension \( n \), the space, \( J^k_x(M, \mathbb{R}) \), of jets at \( x \in M \) of order \( k \), is a Weil algebra of dimension \( C^{k}_{n+k} \) and of order \( k \).

5. If \( A \) is a Weil algebra with maximal ideal \( m_A \) and of order \( h \) and \( B \) a Weil algebra with maximal ideal \( m_B \) and of order \( l \), then the tensor product \( A \otimes B \) is a Weil algebra of maximal ideal \( m_{A \otimes B} = m_A \otimes B + A \otimes m_B \) and of order \( h + l \). Thus, \( \mathbb{D} \otimes \mathbb{D} = \mathbb{R}[T_1, T_2]/(T_1^2, T_2^2) \) is a Weil algebra of order 2.

Remark 1 The tensor product of two algebras of truncated polynomials is not a truncated polynomials algebra. This is the case for \( \mathbb{D} \otimes \mathbb{D} \).

If \( M \) is a smooth manifold, \( C^\infty(M) \) the algebra of real functions on \( M \) and \( A \) a Weil algebra with maximal ideal \( m \), a near point of \( x \in M \) of kind \( A \) is a homomorphism of algebras

\[
\xi: C^\infty(M) \longrightarrow A
\]

such that \([\xi(f) - f(x)] \in m\) for every \( f \in C^\infty(M)[10]\).

Let \( M_x^A \) denote the set of near points of \( x \) of kind \( A \) and

\[
M^A = \bigcup_{x \in M} M_x^A.
\]

The set \( M^A \) is a smooth manifold of dimension \( \dim M \times \dim A \): its the manifold of near points of \( M \) of kind \( A \) or simply the Weil bundle over \( M \) of kind \( A \).

Example 2: For any smooth manifold \( M \), \( M^\mathbb{R} = M \).
1. For any smooth manifold \( M \), the map
\[
TM \longrightarrow \text{Hom}_{A \mathbb{R}}(C^\infty(M), \mathbb{D}), \ v \longmapsto \xi_v,
\]
defined by
\[
\xi_v(f) = f(p) + v(f) \cdot \varepsilon
\]
if \( v \in T_pM \), identifies \( TM = J^1_0(\mathbb{R}, M) \) at \( M^\mathbb{D} \). We verify that \( v \) is a tangent vector at \( p \in M \) if only if \( \xi_v \) is a near point of \( p \in M \) of kind \( \mathbb{D} \).

2. If \( A = \mathbb{R}[X]/(X^3) \), \( M^A = J^2_0(\mathbb{R}, M) \). More generally, if \( A \) is the algebra of truncated polynomials
\[
\mathbb{R}[X_1, \ldots, X_s]/(X_1, \ldots, X_s)^{k+1},
\]
then \( M^A = J^k_0(\mathbb{R}^s, M) \) is the set of jets at 0 of order \( k \) of differentiable applications from \( \mathbb{R}^s \) in \( M \).

3. The application \( \xi \longmapsto \xi(id_{\mathbb{R}}) \) identifies \( \mathbb{R}^A \) to \( A \).

4. If \( V \) is a real vector space of finite dimension, if \((e_i)_{i=1,\ldots,r}\) is a basis of \( V \) and if \((e^*_i)_{i=1,\ldots,r}\) is a dual basis of the basis \((e_i)_{i=1,\ldots,r}\), then
\[
V^A \longrightarrow V \otimes A, \xi \longmapsto \sum_{i=1}^r e_i \otimes \xi(e^*_i)
\]
is a canonical isomorphism of \( A \)-modules.

When \( M \) and \( N \) are two manifolds, and when
\[
h : M \longrightarrow N
\]
is a differentiable application of class \( C^\infty \), then the application
\[
h^A : M^A \longrightarrow N^A, \xi \longmapsto h^A(\xi),
\]
such that for all \( g \in C^\infty(N) \),
\[
[h^A(\xi)](g) = \xi(g \circ h),
\]
is differentiable of class \( C^\infty \). When \( h \) is a diffeomorphism, it is even \( h^A \).

Furthermore, if \( \varphi : A \longrightarrow B \) is a homomorphism of Weil algebras, for any smooth manifold \( M \), the application
\[
\varphi_M : M^A \longrightarrow M^B, \xi \longmapsto \varphi \circ \xi
\]
is differentiable. In particular, the augmentation
\[
A \longrightarrow \mathbb{R}
\]
defined for any smooth manifold \( M \), the projection
\[
M^A \longrightarrow M,
\]
which associates to each near point of \( x \in M \), its origin \( x \).
2 Parallelism of Weil bundles

In what follows $M$ is a smooth manifold of dimension $n$, $A$ a Weil algebra with unit $1_A$, $C^\infty(M)$ the algebra of real functions of class $C^\infty$ on $M$, $\mathfrak{X}(M)$ the $C^\infty(M)$-module of vector fields on $M$, $TM$ the tangent bundle of $M$ and

$$\pi_M : TM \longrightarrow M$$

the canonical projection.

If $(U, \varphi)$ is a local chart of $M$ with local coordinates $(x_1, x_2, \ldots, x_n)$, the application,

$$U^A \longrightarrow A^n, \xi \mapsto (\xi(x_1), \xi(x_2), \ldots, \xi(x_n)),$$

is a bijection from $U^A$ to a open of $A^n$. The manifold $M^A$ is modeled on $A^n$, i.e $M^A$ is a $A$-manifold of dimension $n$.

The set, $C^\infty(M^A, A)$ of the functions of class $C^\infty$ on $M^A$ with values in $A$, is a commutative $A$-algebra with unit. By identifying $\mathbb{R}^A$ at $A$, for $f \in C^\infty(M)$, the application

$$f^A : M^A \longrightarrow A, \xi \mapsto \xi(f),$$

is of class $C^\infty$. Futher the application

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A,$$

is a injective homomorphism of algebras and we have:

$$(f + g)^A = f^A + g^A; \quad (\lambda \cdot f)^A = \lambda \cdot f^A; \quad (f \cdot g)^A = f^A \cdot g^A$$

with $\lambda \in \mathbb{R}$, $f$ and $g$ belonging to $C^\infty(M)$.

When $(a_\alpha)_{\alpha=1,2,\ldots,\dim(A)}$ is a basis of $A$ and when $(a^*_\alpha)_{\alpha=1,2,\ldots,\dim(A)}$ is a dual basis of the basis $(a_\alpha)_{\alpha=1,2,\ldots,\dim(A)}$, the application

$$\sigma : C^\infty(M^A, A) \longrightarrow A \otimes C^\infty(M^A), \varphi \longmapsto \sum_{\alpha=1}^{\dim(A)} a_\alpha \otimes (a^*_\alpha \circ \varphi),$$

is an isomorphism of $A$-algebras. This isomorphism does not depend on the chosen basis and the application

$$\gamma : C^\infty(M) \longrightarrow A \otimes C^\infty(M^A), f \longmapsto \sigma(f^A),$$

is a morphism of algebras.

We note $\mathfrak{X}(M^A)$, the set of all vector fields on $M^A$. The following assertions are then equivalent [4]:

$$\text{[ ]}$$
1. \( X : C^\infty(M^A) \to C^\infty(M^A) \) is a vector field on \( M^A \);

2. \( X : C^\infty(M) \to C^\infty(M^A, A) \) is a linear application satisfying

\[
X(fg) = X(f) \cdot g^A + f^A \cdot X(g)
\]

for all \( f \) and \( g \) in \( C^\infty(M) \).

Then, when

\[
\theta : C^\infty(M) \to C^\infty(M)
\]

is a vector field over \( M \), the application

\[
\theta^A : C^\infty(M) \to C^\infty(M^A, A), f \mapsto [\theta(f)]^A,
\]

is a vector field on \( M^A \): the vector field \( \theta^A \) is the prolongation to \( M^A \) of the vector field \( \theta \).

When \( X \) is a vector field on \( M^A \), considered as derivation from \( C^\infty(M) \) to \( C^\infty(M^A, A) \), then there exists a unique derivation [4],

\[
\tilde{X} : C^\infty(M^A, A) \to C^\infty(M^A, A)
\]

such that:

1. \( \tilde{X} \) is \( A \)-linear;

2. \( \tilde{X} \left[ C^\infty(M^A) \right] \subset C^\infty(M^A) \);

3. \( \tilde{X}(f^A) = X(f) \) for any \( f \in C^\infty(M) \).

The set \( \mathfrak{X}(M^A) \) of vector fields on \( M^A \) is in these conditions, a \( C^\infty(M^A, A) \)-module and a Lie algebra over \( A \) [4].

**Theorem 1** [10]. If \( M \) is a smooth manifold and if \( A \) and \( B \) are two Weil algebras, then the application

\[
(M^A)^B \to M^{A \otimes B}, \eta \mapsto (id_A \otimes \eta) \circ \gamma
\]

is a diffeomorphism.

In particular, we have an isomorphism of manifolds between \( TM^A \) and \( (TM)^A \).

For \( x \in M \), \( T_xM \) denotes the tangent space at \( x \) to \( M \).
We recall that the manifold $M$ is parallelizable if its tangent bundle $TM$ is trivial, i.e., there is a diffeomorphism

$$\sigma : TM \longrightarrow M \times \mathbb{R}^n$$

such that the following diagram commutes and that for all $x \in M$ the restriction

$$\sigma|_{T_xM} : T_xM \longrightarrow \{x\} \times \mathbb{R}^n$$

is an isomorphism of vector spaces.

When $(U, \varphi)$ is a local chart of $M$ with the local coordinates $(x_1, x_2, \ldots, x_n)$, the application

$$\psi : TU^A \longrightarrow U^A \times A^n, \sum_{i=1}^n \lambda_i \cdot \left( \frac{\partial}{\partial x_i} \right)^A |_\xi \longmapsto (\xi, \lambda_1, \ldots, \lambda_n)$$

is a diffeomorphism of $A$-manifolds satisfying $pr_1 \circ \psi = \pi_{M^A}$.

Thus the local parallelism of $M^A$ expressed in terms of existence of a diffeomorphism of $A$-manifolds whose restriction in each tangent space is an isomorphism of $A$-modules.

The aim of this work is to give the equivalence of parallelism of $M^A$ in terms of $A$-manifolds. We recall that when $M$ is a smooth manifold, the basic algebra of $M$ is $C^\infty(M)$. Since $\mathcal{X}(M^A)$ is a $C^\infty(M^A, A)$-module, considered as the set of derivations from $C^\infty(M)$ to $C^\infty(M^A, A)$, and is a Lie algebra over $A$, and as $M^A$ is an $A$-manifold, this means that the basic algebra of $M^A$ is $C^\infty(M^A, A)$ and not $C^\infty(M^A)$.

**Proposition 2** The manifold $M^A$ is parallelizable if and only if there is a diffeomorphism of $A$-manifolds

$$H : TM^A \longrightarrow M^A \times A^n$$

such that the following diagram

$$\begin{array}{ccc}
TM^A & \xrightarrow{H} & M^A \times A^n \\
\downarrow \pi_{M^A} & & \downarrow pr_1 \\
M^A & & \\
\end{array}$$
commutes and that for every \( \xi \in M^A \), the restriction
\[
H_{|T_\xi M^A} : T_\xi M^A \longrightarrow \{\xi\} \times A^n
\]
is an isomorphism of \( A \)-modules.

**Proof.** \( / \implies \) As the manifold is parallelizable, there exists a diffeomorphism
\[
TM^A \xrightarrow{\sigma} M^A \times \mathbb{R}^{n \cdot \dim A}
\]
such that
\[
\begin{array}{ccc}
TM^A & \xrightarrow{\sigma} & M^A \times \mathbb{R}^{n \cdot \dim A} \\
\downarrow \pi_{M^A} & & \downarrow \text{pr}_1 \\
M^A
\end{array}
\]
commutes i.e \( \text{pr}_1 \circ \sigma = \pi_{M^A} \), and that for every \( \xi \in M^A \), the restriction
\[
\sigma_{|T_\xi M^A} : T_\xi M^A \longrightarrow \{\xi\} \times \mathbb{R}^{n \cdot \dim A}
\]
is an isomorphism of vector spaces.

Let
\[
h : A^n \longrightarrow \mathbb{R}^{n \cdot \dim A}
\]
be an isomorphism of vector spaces. By transport of structure, we equip \( \mathbb{R}^{n \cdot \dim A} \) of a structure of \( A \)-module defined on \( A^n \). Thus \( h \) becomes an isomorphism of \( A \)-modules. In the same way,
\[
\sigma_{|T_\xi M^A} : T_\xi M^A \longrightarrow \{\xi\} \times \mathbb{R}^{n \cdot \dim A}
\]
becomes an isomorphism of \( A \)-modules. Asking \( H = (id_{M^A} \times h^{-1}) \circ \sigma \). For every \( \xi \in M^A \), we deduce that the restriction
\[
H_{|T_\xi M^A} : T_\xi M^A \longrightarrow \{\xi\} \times A^n
\]
is an isomorphism of \( A \)-modules.

Since \( \sigma \) is differentiable on a open of \( \mathbb{R}^{2n \cdot \dim A} \), it is the same for \( H \): thus the differentiability of \( H \) carried on open of \( A^n \).

\( \implies / \) The sufficient condition is obvious. ■

The description of the parallelism of \( M^A \) is the following:

**Theorem 3** If \( M \) is a smooth manifold of dimension \( n \) and if \( M^A \) is the Weil bundle of \( M \) of kind \( A \), then the following assertions are equivalent:

1. The manifold \( M^A \) is parallelizable;
2. There are \( n \) vector fields \( X_1, \ldots, X_n \) on \( M^A \) such that for every \( \xi \in M^A \), the vectors \( X_1(\xi), \ldots, X_n(\xi) \) provide a basis of \( T_\xi M^A \);

3. The \( C^\infty(M^A, A) \)-module, \( \mathfrak{X}(M^A) \), of vector fields on \( M^A \) is a free \( C^\infty(M^A, A) \)-module of rank \( n \).

**Proof.** Show 1/ \( \iff \) 2/

1/ \( \implies \) 2/ As the manifold \( M^A \) est parallelizable, then there exists a diffeomorphism of \( A \)-manifolds

\[
H : TM^A \longrightarrow M^A \times A^n
\]
such that the following diagram

\[
\begin{array}{ccc}
TM^A & \xrightarrow{H} & M^A \times A^n \\
\downarrow \pi_M^A & & \downarrow \text{pr}_1 \\
M^A & & \\
\end{array}
\]

commutes and for every \( \xi \in M^A \), the restriction

\[
H|_{T_\xi M^A} : T_\xi M^A \longrightarrow \{\xi\} \times A^n
\]
is an isomorphism of \( A \)-modules.

For all \( i = 1, 2, \ldots, n \), let \( a_i = (0, 0, \ldots, 1_A, 0, \ldots, 0) \) where \( 1_A \) is to \( i \)-th place. Obviously \( (a_1, a_2, \ldots, a_n) \) is a basis of the \( A \)-module \( A^n \). For all \( i = 1, 2, \ldots, n \), the applications

\[
\sigma_i : M^A \longrightarrow M^A \times A^n, \xi \longmapsto (\xi, a_i)
\]

and

\[
X_i = H^{-1} \circ \sigma_i : M^A \longrightarrow TM^A
\]

are differentiables. Moreover

\[
X_i : M^A \longrightarrow TM^A
\]
is a section of the tangent bundle, since for \( \xi \in M^A \) we have

\[
(\pi_M^A \circ X_i)(\xi) = (\text{pr}_1 \circ H) \left[ (H^{-1} \circ \sigma_i)(\xi) \right] \\
= (\text{pr}_1 \circ H) \left[ H^{-1}(\xi, a_i) \right] \\
= \text{pr}_1(\xi, a_i) \\
= \xi.
\]
Thus
\[ \pi_{M^A} \circ X_i = \text{id}_{M^A}. \]

We conclude that \( X_i \) is a vector field on \( M^A \).

For any \( \xi \in M^A \), as \( (\xi, a_i)_{i=1,2,...,n} \) is a basis of the \( A \)-module \( \{\xi\} \times A^n \), then \( [H^{-1}(\xi, a_i)]_{i=1,2,...,n} \) is a basis of the \( A \)-module \( T_\xi M^A \). We conclude that the vectors \( X_1(\xi), ..., X_n(\xi) \) form a basis of the \( A \)-module \( T_\xi M^A \).

\[ 2/ \implies 1/ \] Assume that there are \( n \) vector fields \( X_1, ..., X_n \) on \( M^A \) such that for \( \xi \in M^A \), \( (X_1(\xi), ..., X_n(\xi)) \) is a basis of \( T_\xi M^A \).

The application
\[ \varphi : M^A \times A^n \longrightarrow TM^A, (\xi, \lambda_1, ..., \lambda_n) \longmapsto \sum_{i=1}^{n} \lambda_i \cdot X_i(\xi) \]
is an diffeomorphism of \( A \)-manifolds and

\[ \varphi|_{\{\xi\} \times A^n} : \{\xi\} \times A^n \longrightarrow T_\xi M^A, (\xi, \lambda_1, ..., \lambda_n) \longmapsto \sum_{i=1}^{n} \lambda_i X_i(\xi) \]
is an isomorphism of \( A \)-modules and its reciprocal

\[ \varphi^{-1} : TM^A \longrightarrow M^A \times A^n, \sum_{i=1}^{n} \lambda_i X_i(\xi) \longmapsto (\xi, \lambda_1, ..., \lambda_n) \]
is such that

\[ pr_1 \circ \varphi^{-1} = pr_1 \circ H = \pi_{M^A}. \]

Then concludes that the manifold \( M^A \) is parallelizable.

Show \( 2/ \iff 3/ \)

\[ 2/ \implies 3/ \] Assume that there are \( n \) vector fields \( X_1, ..., X_n \) on \( M^A \) such that for \( \xi \in M^A \), \( (X_1(\xi), ..., X_n(\xi)) \) is a basis of \( T_\xi M^A \).

The vector fields \( X_1, ..., X_n \) are linearly independent. Indeed, if, \( g_1, ..., g_n \in C^\infty(M^A, A) \) are such that

\[ \sum_{i=1}^{n} g_i \cdot X_i = 0, \]

then for all \( \xi \in M^A \), we have

\[ \sum_{i=1}^{n} g_i(\xi) \cdot X_i(\xi) = 0. \]
As \((X_1(\xi), ..., X_n(\xi))\) is a basis of \(T_\xi M^A\), then \(g_i(\xi) = 0\) for all \(i = 1, 2, ..., n\). As \(\xi\) is arbitrary, we conclude that \(g_i = 0\) for all \(i = 1, 2, ..., n\).

The family \(X_1, ..., X_n\) generates \(\mathfrak{X}(M^A)\), in fact, if \(Y \in \mathfrak{X}(M^A)\) and \(\xi \in M^A\), we have:

\[
Y(\xi) = \sum_{i=1}^{n} \lambda_i \cdot X_i(\xi)
\]

with the \(\lambda_i \in A\).

The application

\[
M^A \xrightarrow{Y} TM^A \xrightarrow{H} M^A \times A^n \xrightarrow{pr_2} A^n \xrightarrow{pr_i} A, \xi \mapsto \lambda_i,
\]

is differentiable. Asking \(f_i = pr_1 \circ pr_2 \circ H \circ Y\), we have \(f_i(\xi) = \lambda_i\) and

\[
Y(\xi) = \sum_{i=1}^{n} f_i(\xi) \cdot X_i(\xi)
\]

\[
= \left( \sum_{i=1}^{n} f_i \cdot X_i \right)(\xi).
\]

As \(\xi\) is arbitrary, then

\[
Y = \sum_{i=1}^{n} f_i \cdot X_i.
\]

Thus \(X_1, ..., X_n\) is a basis of the \(C^\infty(M^A, A)\)-module \(\mathfrak{X}(M^A)\). We conclude that \(\mathfrak{X}(M^A)\) is a free \(C^\infty(M^A, A)\)-module of rank \(n\).

3/ \implies 2/ Assume that \(\mathfrak{X}(M^A)\) is a free \(C^\infty(M^A, A)\)-module of rank \(n\). Let \((X_1, ..., X_n)\) a basis of the \(C^\infty(M^A, A)\)-module \(\mathfrak{X}(M^A)\).

If \(\alpha_1(\xi), ..., \alpha_n(\xi)\) are elements of \(A\) such that

\[
\sum_{i=1}^{n} \alpha_i(\xi) \cdot X_i(\xi) = 0
\]

for all \(\xi \in M^A\), and for all \(i = 1, 2, ..., n\), let

\[
f_i : M^A \longrightarrow A, \xi \mapsto \alpha_i(\xi).
\]

For \(\eta \in M^A\), there exists \(Y \in \mathfrak{X}(M^A)\) such that \(Y(\eta) = \sum_{i=1}^{n} f_i(\eta) \cdot X_i(\eta)\). As \(Y\) is differentiable in a neighborhood of \(\eta\), it is even \(f_i\) in a neighborhood of \(\eta\).
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Since \( \eta \) is arbitrary, we deduce that the \( f_i \) are differentiable. Thus, we have

\[
0 = \sum_{i=1}^{n} \alpha_i(\xi) \cdot X_i(\xi) \\
= \sum_{i=1}^{n} f_i(\xi) \cdot X_i(\xi) \\
= \left( \sum_{i=1}^{n} f_i \cdot X_i \right)(\xi)
\]

for any \( \xi \in M^A \). As the vector fields \( X_1, ..., X_n \) form a basis of the \( C^\infty(M^A, A) \)-module \( \mathfrak{X}(M^A) \), then

\[
f_1 = ... = f_n = 0
\]
i.e that for all \( \xi \in M^A \), the \( \alpha_i(\xi) = 0 \). It is concluded that the family \( (X_1(\xi), ..., X_n(\xi)) \) is free for all \( \xi \in M^A \).

Moreover, for \( v \in T_\xi M^A \), there exists a vector field \( Y \in \mathfrak{X}(M^A) \) such that \( Y(\xi) = v \). Since

\[
Y = \sum_{i=1}^{n} f_i \cdot X_i
\]

where each \( f_i \in C^\infty(M^A, A) \), then,

\[
v = \sum_{i=1}^{n} f_i(\xi) \cdot X_i(\xi).
\]

Thus, the family \( (X_1(\xi), ..., X_n(\xi)) \) generates the \( A \)-module \( T_\xi M^A \).

Then concludes that at every \( \xi \in M^A \), the vectors \( X_1(\xi), ..., X_n(\xi) \) form a basis of the \( A \)-module \( T_\xi M^A \).  

\textbf{Corollary 4} If \( M \) is a parallelizable manifold, then the manifold \( M^A \) is parallelizable.

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\textbf{References}


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