Stone Spaces and Compactifications

Monerah Al-Hajri

University of Dammam, College of Sciences
P.O. Box 838, 31113 Dammam, KSA
M3sbkh@yahoo.com

Karim Belaid

University of Dammam, College of Sciences
P.O. Box 838, 31113 Dammam, KSA
belaid412@yahoo.fr

Othman Echi

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
PO Box 5046, Dhahran 31261, KSA
echi@kfupm.edu.sa

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Abstract

This paper deal with spaces such that their compactification is a Stone space. The particular cases of the one-point compactification, the Wallman compactification and the Stone-Čech compactification are studied.

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1 Introduction

A topological space $X$ is called a zero dimensional if $X$ has a basis consisting of clopen sets. In Bourbaki [1] the condition Hausdorff is added for a space $X$
to be a zero dimensional space.

A totally disconnected space is a topological space that is maximally disconnected, in the sense that it has no non-trivial connected subsets. Hence a topological space $X$ is a totally disconnected space if and only if for any $x \neq y$ in $X$, there exists a clopen set $U$ of $X$ such that $x \in U$ and $y \notin U$. The concepts of zero-dimensionality and total disconnectedness are closely related. Indeed, every zero-dimensional $T_1$-space is totally disconnected.

A Stone space, also called a Boolean space, is a topological space that is zero-dimensional, $T_0$ and compact. Equivalently, a Stone space is a totally disconnected compact space.

Recall that a compactification of a topological space $X$ is a couple $(K(X), e)$, where $K(X)$ is a compact space and $e : X \to K(X)$ is a continuous embedding ($e$ is a continuous one-to-one map and induces a homeomorphism from $X$ onto $e(X)$) such that $e(X)$ is a dense subspace of $K(X)$. When a compactification $(K(X), e)$ of $X$ is given, $X$ will be identified with $e(X)$ and assumed to be dense in $K(X)$.

The first section of this paper contains some remarks and properties of clopen sets on a compactification $K(X)$ of a space $X$.

The second section deals with a characterization of space such that its one point compactification (resp. Wallman compactification) is a Stone space.

In the third section we give a characterization of space $X$ such that $\beta(\rho(X))$ is a Stone space with $\rho(X)$ is the universal Tychonoff space and $\beta(\rho(X))$ is the Stone-Čech compactification of $\rho(X)$.

## 2 Clopen sets and compactifications

Our goal in the present section is to give some useful observation about clopen sets on a compactification $K(X)$ of a topological space $X$.

**Proposition 2.1** Let $X$ be a topological space and $K(X)$ be a compactification of $X$. Let $F$ be a closed set of $K(X)$ and $O$ be an open set of $K(X)$. If $F \cap X = O \cap X$ then $O \subseteq F$.

**Proof 2.2** Suppose that $O \notin F$. Then $O - F$ is a nonempty open set of $K(X)$. Since $F \cap X = O \cap X$, $O - F \subseteq K(X) - X$ contradicting the fact that $X$ is a dense set of $K(X)$.

**Corollary 2.3** If $L$ is a clopen set of $X$ such that there exist two clopen sets $H_1$ and $H_2$ of $K(X)$ with $L = H_1 \cap X = H_2 \cap X$, then $H_1 = H_2$.

**Proposition 2.4** Let $X$ be a topological space and $K(X)$ be a compactification of $X$. A subset $L$ of $X$ is a clopen if and only if $L = X \cap \text{cl}_{K(X)}(L) = X \cap \text{int}(\text{cl}_{K(X)}(L))$. 
Proof 2.5. Necessary condition. Since $L$ is a closed set of $X$, there exists a closed set $C$ of $K(X)$ such that $C \cap X = L$. Hence $L \subseteq cl_{K(X)}(L) \subseteq C$, so $cl_{K(X)}(L) \cap X = L$. Let $O$ be an open set of $K(X)$ such that $O \cap X = L$. By Proposition 2.1, $O \subseteq cl_{K(X)}(L)$. Thus $O \subseteq int(cl_{K(X)}(L))$. Since $O \subseteq int(cl_{K(X)}(L)) \subseteq cl_{K(X)}(L)$ and $O \cap X = cl_{K(X)}(L) \cap X = L$, $int(cl_{K(X)}(L)) \cap X = L$. Therefore $L = X \cap cl_{K(X)}(L) = X \cap int(cl_{K(X)}(L))$.

Proposition 2.6. Let $X$ be a topological space, $L$ be a clopen set of $X$ and $K(X)$ be a compactification of $X$. If there exists a clopen set $H$ of $K(X)$ such that $H \cap X = L$, then $H = cl_{K(X)}(L)$.

Proof 2.7. As $H$ is an open set of $K(X)$, $H \subseteq cl_{K(X)}(L)$ (by Proposition 2.1). That $cl_{K(X)}(L) \subseteq H$ is due to the fact that $H$ is a closed set containing $L$.

Let $X$ be a non compact topological space, set $\tilde{X} = X \cup \{\infty\}$ with the topology whose members are the open sets of $X$ and all subsets $U$ of $\tilde{X}$ such that $\tilde{X} \setminus U$ is a closed compact set of $X$. The space $\tilde{X}$ is called the one-point compactification of $X$ (or the Alexandroff compactification of $X$).

Proposition 2.8. Let $X$ be a non-compact topological space and $L$ be a clopen of $X$. Then there exists a clopen set $H$ of the Alexandroff compactification $\tilde{X}$ such that $H \cap X = L$ if and only if either $L$ or $X - L$ is compact.

Proof 2.9. Necessary condition.
If $\infty \in H$, then $\tilde{X} - H$ is a closed set of $\tilde{X}$ not containing $\infty$. Hence $X - L$ is compact.
If $\infty \notin H$, then $H$ is a closed set of $\tilde{X}$ not containing $\infty$. Hence $L$ is compact.

Sufficient condition. It is immediate that if $L$ is compact, then $L$ is a clopen set of $\tilde{X}$. If $X - L$ is a compact set of $X$, then $X - L$ is a clopen set of $\tilde{X}$. So $L \cup \{\infty\}$ is a clopen set of $\tilde{X}$ such that $(L \cup \{\infty\}) \cap X = L$.

Proposition 2.10. Let $X$ be a topological space and $K(X)$ be a compactification of $X$. If $K(X)$ is a Stone space, then the following properties hold:

1. $X$ is totally disconnected.
2. For each open set $O$ of $X$ and $x \in O$ there exists a clopen set $C$ such that $x \in C \subseteq O$.

Proof 2.11. (1) Straightforward.
(2) Let $O$ be an open set of $X$ and $x \in O$. Then there exists an open set $U$ of $K(X)$ such that $O = U \cap X$. Clearly, $K(X) - U$ is a compact closed set of
K(X), and for each \( y \in K(X) - U \) there exits a clopen set \( C_y \) of \( K(X) \) such that \( x \in C_y \) and \( y \notin C_y \). So there exists a finite subset \( Y \) of \( K(X) - U \) such that \( K(X) - U \subseteq \bigcup \{ K(X) - C_y : y \in Y \} \). Hence \( C = \cap \{ C_y : y \in Y \} \) is a clopen neighborhood of \( x \) such that \( C \subseteq U \). Thus \( C \cap X \) is a clopen set of \( X \) and \( x \in C \cap X \subseteq O \).

3 Space such that its one-point compactification (resp. Wallman compactification) is a Stone space

**Proposition 3.1** Let \( X \) be a topological space. Then the one-point compactification \( \tilde{X} \) of \( X \) is a Stone space if and only if \( X \) is a \( T_2 \)-space and the collection of compact clopen sets is a base of \( X \).

**Proof 3.2** Necessary condition. Since \( \tilde{X} \) is a Stone space, \( X \) is \( T_2 \) and \( \tilde{X} \) has a basis consisting of clopen sets. Let \( U \) be an open set of \( X \). So \( U \) is an open set of \( \tilde{X} \). Hence there exits a collection \( \mathcal{O} \) of compact clopen sets of \( \tilde{X} \) such that \( U = \bigcup \{ O : O \in \mathcal{O} \} \). But, for each \( O \in \mathcal{O} \), \( O \subseteq X \); so \( O \) is a compact clopen set of \( X \). Thus the collection of compact clopen sets is a base of \( X \).

Sufficient condition. First, remark that if \( U \) is a compact closed set of \( X \), then \( U \) is a compact closed set of \( \tilde{X} \).

Let \( x \neq y \) be in \( X \). We consider two cases.

Case 1. \( x, y \in X \). Since \( X \) is a \( T_2 \)-space, there exists a compact clopen set \( U \) of \( X \) such that \( x \in U \) and \( y \notin U \). To complete the proof this case it suffices to remark that \( U \) is also a compact closed set of \( \tilde{X} \).

Case 2. \( x \in X \) and \( y = \infty \). Since the collection of compact clopen sets is a base of \( X \), there exists a compact clopen set \( U \) of \( X \) such that \( x \in U \). Therefore \( \tilde{X} \) is a Stone space, since \( U \) is a compact clopen set of \( \tilde{X} \).

Recall that the Wallman compactification of \( T_1 \)-space was introduced, in 1938 [4], by Wallman as follow:

Let \( \mathcal{P} \) be a class of subsets of a topological space \( X \) which is closed under finite intersections and finite unions.

A \( \mathcal{P} \)-filter on \( X \) is a collection \( \mathcal{F} \) of nonempty elements of \( \mathcal{P} \) with the properties:

(i) \( \mathcal{F} \) is closed under finite intersections;

(ii) \( P_1 \in \mathcal{F}, P_1 \subseteq P_2 \) implies \( P_2 \in \mathcal{F} \).
A \mathcal{P}\text{-ultrafilter} is a maximal \mathcal{P}\text{-filter. When } \mathcal{P} \text{ is the class of closed sets of } X, \text{ then the } \mathcal{P} \text{-filters are called closed filters.}

The points of the Wallman compactification \(wX\) of a space \(X\) are the closed ultrafilters on \(X\). For each closed set \(D \subseteq X\), define \(D^*\) to be the set \(D^* = \{A \in wX \mid D \in A\}\). Thus \(\mathcal{C} = \{D^* \mid D \text{ is a closed set of } X\}\) is a base for the closed sets of a topology on \(wX\). Let \(U\) be an open set of \(X\), we define \(U^* = \{A \in wX \mid F \subseteq U \text{ for some } F \in A\}\), it is easily seen that the class \(\{U^* \mid U\text{ is an open set of } X\}\) is a base for open sets of the topology of \(wX\), and the following properties of \(wX\) are frequently useful:

(i) If \(U \subseteq X\) is open, then \(wX - U^* = (X - U)^*\).

(ii) If \(D \subseteq X\) is closed, then \(wX - D^* = (X - D)^*\).

(iii) If \(U_1\) and \(U_2\) are open sets of \(X\), then \((U_1 \cap U_2)^* = U_1^* \cap U_2^*\) and \((U_1 \cup U_2)^* = U_1^* \cup U_2^*\).

**Proposition 3.3** Let \(X\) be a \(T_1\)-space. Then \(wX\) is a Stone space if and only if for each disjoint closed sets \(F\) and \(G\) of \(X\), there exists a clopen set \(U\) such that \(F \subseteq U\) and \(G \cap U = \emptyset\).

**Proof 3.4** Necessary condition. First, remark that if \(Q\) is a clopen set of \(wX\), then there exists a clopen set \(U\) of \(X\) such that \(Q = U^*\). In fact, let \(\mathcal{V}\) be a collection of open sets of \(X\) such that \(Q = \cup\{V^* : V \in \mathcal{V}\}\). Since \(Q\) is a closed set of \(wX\), \(Q\) is a compact set of \(wX\). Hence there exists a finite subcollection \(\mathcal{V}'\) of \(\mathcal{V}\) such that \(Q = \cup\{V'^* : V \in \mathcal{V}'\}\). Thus \(Q = U^*\) with \(U = \cup\{V : V \in \mathcal{V}'\}\). That \(U\) is a clopen set of \(X\) follows immediately from the fact that \(U = Q \cap X\).

Let \(F\) and \(G\) be two disjoint closed sets of \(X\). Then \(F^*\) and \(G^*\) are two disjoint closed sets of \(wX\). Let \(\mathcal{F} \in F^*\) and \(\mathcal{G} \in G^*\). Since \(wX\) is a Stone space, there exists a clopen set \(U\) of \(X\) such that \(\mathcal{F} \in U^*\) and \(\mathcal{G} \notin U^*\). So, for each \(\mathcal{G} \in G^*\), there exists a collection \(\mathcal{V}\) of clopen sets of \(X\) such that \(F^* \subseteq \cup\{V^* : V \in \mathcal{V}\}\) and \(\mathcal{G} \notin \cup\{V^* : V \in \mathcal{V}\}\). Since \(F^*\) is a compact closed set of \(wX\), there exists a finite subcollection \(\mathcal{V}'\) of \(\mathcal{V}\) such that \(F^* \subseteq \cup\{V'^* : V \in \mathcal{V}'\}\) and \(\mathcal{G} \notin \cup\{V'^* : V \in \mathcal{V}'\}\). Set \(W_{\mathcal{G}} = \cap\{wX - V'^* : V \in \mathcal{V}'\}\). Hence \(W_{\mathcal{G}}\) is a clopen neighborhood of \(\mathcal{G}\) such that \(W_{\mathcal{G}} \cap F^* = \emptyset\). Set \(\mathcal{W} = \{W_{\mathcal{G}} : \mathcal{G} \in G^*\}\). Since \(G^*\) is a compact set of \(wX\), there exists a finite subcollection \(\mathcal{W}'\) of \(\mathcal{W}\) such that \(G^* \subseteq \cup\{W_{\mathcal{G}} : W_{\mathcal{G}} \in \mathcal{W}'\}\). Thus \(Q = \cup\{W_{\mathcal{G}} : W_{\mathcal{G}} \in \mathcal{W}'\}\) is a clopen set of \(wX\) such that \(G^* \subseteq Q\) and \(F^* \subseteq wX - Q\). Then there exists a clopen set \(U\) of \(X\) such that \(U^* = wX - Q\). Therefore \(F \subseteq U\) and \(G \cap U = \emptyset\).

Sufficient condition. Let \(\mathcal{F}\) and \(\mathcal{G}\) two distinct element of \(wX\). Then there exist two closed sets \(F\) and \(G\) of \(X\) such that \(F \in \mathcal{F}\), \(G \in \mathcal{G}\) and \(F \cap G = \emptyset\). Hence there exists a clopen set \(U\) of \(X\) such that \(F \subseteq U\) and \(G \cap U = \emptyset\). Thus \(F^* \subseteq U^*\) and \(G^* \cap U^* = \emptyset\). Since \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\), \(\mathcal{F} \in F^*\) and \(\mathcal{G} \in G^*\). So
4 Application

Let $C(X)$ be the ring of all real valued continuous functions defined on $X$.

The construction of the universal Tychonoff space (or $\rho$-identification) of a topological space is as follows:

Let $X$ be a topological space and $\sim$ the equivalence relation on $X$ defined by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in C(X)$.

Let $\rho(X)$ denote the set of equivalence classes and let $\theta : X \rightarrow \rho(X)$ be the canonical onto map assigning to each point of $X$ its equivalence class.

Since every $f$ in $C(X)$ is constant on each equivalence class, we can define $\rho(f) : \rho(X) \rightarrow \mathbb{R}$ by $\rho(f)(\theta(x)) = f(x)$.

Now equip $\rho(X)$ with the topology $T_\rho$ whose closed sets are of the form $\cap \{\rho(f) \circ (\theta) : \alpha \in I\}$, where $f : X \rightarrow \mathbb{R}$ is a continuous map and $F$ a closed set of $\mathbb{R}$. It is well known that, $(\rho(X), T_\rho)$ is a Tychonoff space [3].

The following remark has been given in [2]:

**Remark 4.1** A closed set of $\rho(X)$ is of the form $\widehat{F} = \cap \{\rho(f) \circ (\theta) : f \in F\}$, where $F$ is a subset of $C(X)$.

Using Proposition 3.3, the following lemma is immediate.

**Lemma 4.2** Let $X$ be a topological space. Then $\beta(\rho(X))$ is a Stone space if and only if for each disjoint closed sets $F$ and $H$ of $\rho(X)$, there exists a clopen set $U$ of $\rho(X)$ such that $F \subseteq U$ and $G \cap U = \emptyset$.

**Lemma 4.3** Let $X$ be a topological space. Then $B = \{\widehat{f} = \cup_{x \in X} \theta(x) : f(x) = 0 \mid f \in C(X)\}$ is a base of closed sets of $\rho(X)$.

**Proof 4.4** Let $F$ be a subset of $C(X)$. It is immediate that $\widehat{F} = \cap \{\rho(f) \circ (\theta) : f \in F\} = \cap \{\theta(x) : x \in X \text{ and } \rho(f)(\theta(x)) = 0 \text{ and } f \in F\}$. Hence $\widehat{F} = \cap_{f \in F} (\cup_{x \in X} \theta(x) \mid f(x) = 0\}.

For the next lemma we denote by $F_X$ the set $\cap \{f^{-1}(\{0\}) : f \in F\}$, where $F$ be a subset of $C(X)$. Using Echi and Lazaar terminology [2] a set $F_X$ is called a zero-closed (z-closed, for short) set of $X$. A z-closed set of $X$ is called a zero-clopen (z-clopen for short) set of $X$ if there exists a subsets $G$ of $C(X)$ such that $X - H_X = G_X$.

The following lemma is an immediate consequence of the fact that for $x \in X, x \in H_X \cap G_X, f(x) = 0$, for each $f \in H \cup G$, if and only if $\theta(x) \in \widehat{H} \cap \widehat{G}$.
Lemma 4.5 Let $X$ be a topological space, $H$ and $G$ two subsets of $C(X)$. Then $\hat{H} \cap \hat{G} = \emptyset$ if and only if $H_X \cap G_X = \emptyset$.

Proposition 4.6 Let $X$ be a topological space. Then $\beta(\rho(X))$ is a Stone space if and only if for each two subset $F$ and $G$ of $C(X)$ such that $F_X \cap G_X = \emptyset$, there exists a $z$-clopen set $U_X$ such that $F_X \subseteq U_X$ and $G_X \cap U_X = \emptyset$.

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References


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