The Multiplicative Closed Set S_a

Sarab A. Al-Taha

Department of Mathematics, Faculty of Science
Al Zaytoonah University, Amman Jordan

Copyright © 2013 Sarab A. Al-Taha. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Let R be a commutative ring with identity, let a be a none zero element of R , this paper deals with a new definition for a multiplicative closed set $S_a = \{ a' \in R : a = aa' \}$ of the ring R , we prove some properties of this set. In [1] we give and study the definition of S_1 ideal, in [2] we give and study the definition of S_2 ideal while in this work we introduce the ring Rs_a , the localization of R at S_a . We prove the following result among others, the principal ideal $\langle a \rangle$ is S_1 ideal of R if, and only if the principal ideal $\langle a/a' \rangle$ is S_1 ideal of the ring Rs_a . We also prove that the principal ideal $\langle a \rangle$ is S_2 ideal of R if, and only if the principal ideal $\langle a/a' \rangle$ is S_2 ideal of Rs_a .

Keywords: commutative rings, ideals, multiplicative closed sets, quotient rings

1 Introduction

Let R be a commutative ring with identity, let a be a nonzero element of R , the multiplicative closed set $S_a = \{a' \in R : a = aa'\}$ will serve as our main tool throughout this paper.

Recall that, the principal ideal $I = \langle a \rangle$ is called S_1 ideal if, and only if the following holds:

$$\text{If } ab = a \quad b \in R \quad (1)$$

There exists $a' \in R$, $a' \neq 0$ such that:

$$a'b = 0 \quad [1] \quad (2)$$

And recall that, the principal ideal $I = \langle a \rangle$ is S_2 ideal of R if, and only if the following holds:

$$\text{If } ab = a \quad b \in R, b \neq 1 \quad (3)$$

There exists $a' \in R$, $a' \neq 0$ such that:

$$a'b = a'a \quad [2] \quad (4)$$

In this work we use,

$$\text{ann}(a) = \{ x \in R : ax = 0 \}$$

2 Main Results

In this section we state down the following results:

Theorem (1): Let R be a commutative ring with identity, let a be a nonzero element of R . Then there exists a multiplicative closed set $S_a = \{a' \in R : a = aa'\}$ of the ring R with $\text{ann}(a) \cap S_a = \emptyset$.

Theorem (2): Let R be a commutative ring with identity. Let a and b be any two nonzero elements of R and S_a, S_b, S_{ab} are multiplicative closed sets of the ring R . Then we have the following:

(a): If $\text{ann}(a) = \text{ann}(b)$, then $S_a = S_b$.

(b): $S_a \subset S_{ab}$.

(c): $S_a \cap S_b \subset S_{a-b}$.

Corollary (1):

Let R be a commutative ring with identity, Let a be a nonzero element of R then

$$S_a \subset S_a^n \text{ for each } n \in \mathbb{Z}^+.$$

Theorem (3): Let R be a commutative ring with identity, let a be a nonzero element of R . Then a is idempotent element of the ring R if, and only if a/a' is idempotent element of the ring R_{S_a} .

Theorem (4): Let R be a commutative ring with identity, let a be a nonzero element of R . Then $\langle a \rangle$ is S_1 ideal of the ring R if, and only if $\langle a/a' \rangle$ is S_1 ideal of the ring R_{S_a} .

Theorem (5): Let R be a commutative ring with identity, let a be a nonzero element of R . Then $\langle a \rangle$ is S_2 ideal of the ring R if, and only if $\langle a/a' \rangle$ is S_2 ideal of the ring R_{S_a} .

Theorem (6): Let a and b be any two nonzero elements of R . $\langle a/a' \rangle$ and $\langle b/b' \rangle$ are S_1 ideals of R_{S_a} and R_{S_b} respectively if, and only if $\langle a/a', b/b' \rangle$ is S_1 ideal of the ring $R_{S_a} \oplus R_{S_b}$.

3 Proofs

In this section we prove the main results of this paper.

Proof of theorem (1)

$S_a = \{a' \in R: a = aa'\}$ is multiplicative closed set since

$$(1) 0 \notin S_a$$

$$(2) \text{ If } a', b' \text{ belong to } S_a \text{ then } a' b' \text{ belong to } S_a. \quad [3], [4]$$

This multiplicative closed set contains 1.

Now, to show that $\text{ann}(a) \cap S_a = \emptyset$. (5)

Let $x \in \text{ann}(a) \cap S_a$, this means that $xa = 0 = a$ which is impossible since a is nonzero element.

Example (1): In the ring $\mathbb{Z}_8$, the multiplicative closed set $S_2 = \{1, 5\}$. (6)

Proof (a) of theorem (2):

We have $\text{ann}(a) = \text{ann}(b)$, in order to show that $S_a \subset S_b$, let $x \in S_a$ It follows that $xa = a$.

Clearly, $a(x-1) = 0$ and therefore $(x-1) \in \text{ann}(a) = \text{ann}(b)$. (7)

This implies that $(x-1)b = 0$, using definition of S_b , we get $x \in S_b$. Similarly we can prove that

$S_b \subset S_a$, this end the proof of part (a) of this theorem.

Proof (b) of theorem (2).

To show that $S_a \subset S_{ab}$, let $x \in S_a$, this means that $xa = a$, clearly $xab = ab$,

This implies that $x \in S_{ab}$. (8)

Remark: S_{ab} is not necessarily subset of S_a .

Example (2):

In $\mathbb{Z}_8$, $S_2 = \{1, 5\}$ while $S_4 = \{1, 3, 5, 7\}$, it's clear that S_4 is not subset of S_2 . (9)

Corollary:

$S_a \subset S_a^n$ for $n \in \mathbb{Z}^+$.

Proof (c) of theorem (2):

To show that $S_a \cap S_b \subset S_{a-b}$ where $a \neq b$.

Let $x \in S_a \cap S_b$, this implies that $x \in S_a$ and $x \in S_b$, we get the following:

$xa = a$ and $xb = b$, it follows that $x(a-b) = (a-b)$ and thus $x \in S_{a-b}$. (10)

Proof of theorem (3):

First, assume that a is an idempotent element of the ring R , we want to show that $\langle a/a' \rangle$ is idempotent element of the ring R_{S_a} [5].

We have to show that:

$a/a' \cdot a/a' = a/a'$ this statement is true since $a'aa = a' a' a$. (11)

Now assume that a/a' is an idempotent element of the ring R_{S_a} , this means that

$a/a' \cdot a/a' = a/a'$ which implies that $a^2 = a$ and hence a is idempotent element of the ring R . (12)

Proof of theorem (4):

Assume that $\langle a \rangle$ is S_1 ideal of the ring R . [1]

To show that the principal ideal $\langle a/a' \rangle$ is S_1 ideal of R_{S_a}

let $a/a' \cdot b/b' = a/a'$ where b/b' belong to the ring R_{S_a} .

Using definition of R_{S_a} , there exists $c' \in S_a$ such that $c'aba' = c'a'b'a$ [5]

This implies that $ab = a$, but $\langle a \rangle$ is S_1 ideal of R , therefore there exists $k \in R$ such that $Kb = 0$. Clearly $k/1 \cdot b/b' = 0$, where $b/b' \in R_{S_a}$.

Thus a/a' is S_1 ideal of R_{S_a} . (13)

Conversely, assume that the principal ideal $\langle a/a' \rangle$ is S_1 ideal of R_{S_a} , to show that the principal ideal $\langle a \rangle$ is S_1 ideal of R , let $ab = a$ where b belong to R .

It follows that $a/a' \cdot b/1 = a/a'$.

But $\langle a/a' \rangle$ is S_1 ideal of R_{S_a} , using (2) there exists $k/k' \in R_{S_a}$ such that $k/k' \cdot b/1 = 0$.

This means that there exists $c' \in R_{S_a}$ such that $c'k \cdot b = 0$, hence $\langle a \rangle$ is S_1 ideal of R . (14)

Proof of theorem (5).

Assume that $\langle a \rangle$ is S_2 ideal of R , to show that $\langle a/a' \rangle$ is S_2 ideal of R_{S_a} .

Let $a/a' \cdot b/b' = a/a'$ where b/b' belong to the ring R_{S_a} . (15)

Using definition of R_{S_a} , there exists $c' \in S_a$ such that $c' a' ab = c' a' b' a$

This implies that $ab = a$, but $\langle a \rangle$ is S_2 ideal of R , using (4) [2]

There exists $k \in R$ such that $kb = ka$.

Clearly $k/1 \cdot b/b' = k/1 \cdot a/a'$. (16)

To prove the converse, let $ab = a$.

$a/a' \cdot b/1 = a/a'$. But $\langle a/a' \rangle$ is S_2 ideal of R_{S_a} , using (4)

There exists $k/k' \in R_{S_a}$ such that:

$k/k' \cdot b/1 = k/k' \cdot a/a'$, this means there exists $c' \in S_a$ such that:

$c' k b k' a' = c' k' k a = c' k' k a' a$. Thus

$k^* b = k^* a$ where $k^* = c' k' k a'$. (17)

Thus $\langle a \rangle$ is S_2 ideal.

Proof of theorem (6):

Assume that $\langle a/a' \rangle, \langle b/b' \rangle$ are S_1 ideals of R_{S_a} and R_{S_b} respectively.

To show that $\langle a/a', b/b' \rangle$ is S_1 ideal of $R_{S_a} \oplus R_{S_b}$.

Let $(a/a', b/b')(c/c', d/d') = (a/a', b/b')$ where $(c/c', d/d') \in R_{S_a} \oplus R_{S_b}$.

We get, $a/a' \cdot c/c' = a/a'$, $b/b' \cdot d/d' = b/b'$.

Since $\langle a/a' \rangle, \langle b/b' \rangle$ are S_1 ideal of R_{S_a} and R_{S_b} respectively, using (2)

There exist $k/k' \in R_{S_a}$ and $h/h' \in R_{S_b}$ respectively such that:

$(a/a', b/b')(k/k', h/h') = (0/1, 0/1)$ where $(k/k', h/h') \in R_{S_a} \oplus R_{S_b}$.

Therefore $\langle a/a', b/b' \rangle$ is S_1 ideal of $R_{S_a} \oplus R_{S_b}$.

To prove the converse, assume that $\langle a/a', b/b' \rangle$ is S_1 ideal of $R_{sa} \oplus R_{sb}$.

In order to show that $\langle a/a' \rangle$, $\langle b/b' \rangle$ are S_1 ideals of R_{sa} and R_{sb} respectively, let $a/a'.c/c' = a/a' \ \& \ b/b'.d/d' = b/b'$.

$$\text{This implies that } (a/a', b/b')(c/c', d/d') = (a/a', b/b') \quad (18)$$

Since $\langle a/a', b/b' \rangle$ is S_1 ideal of $R_{sa} \oplus R_{sb}$, using (2)

There exists $(k/k', h/h') \in R_{sa} \oplus R_{sb}$ such that:

$$(c/c', d/d')(k/k', h/h') = (0/1, 0/1) \text{ where } (k/k', h/h') \in R_{sa} \oplus R_{sb}.$$

Directly we get that:

$c/c'.k/k' = 0/1$ and $d/d'.h/h' = 0/1$, this implies that $\langle a/a' \rangle$ and $\langle b/b' \rangle$ are S_1 ideals of R_{sa} and R_{sb} respectively.

REFERENCES

- [1] Sarab A. Al-Taha: S_1 ideals in commutative rings, Jour.of Inst & Computer Sciences Vol. 24. No. 1 (2011). 23-27.
- [2] Sarab A. Al-Taha: S_2 ideals in commutative rings, Journal of Advanced Social Research 1(2011). 58-62.
- [3] T.S.BLYTH o E.F.ROBERTSON: Groups, Rings and fields. Cambridge University Press. (1984).
- [4] Thomas W. Hungerford: Algebra. Springer- Velag (1974).
- [5] Joseph A. Gallian: Contemporary Abstract Algebra. Houghton Mifflin Company (2006).

Received: February 28, 2013