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Quasi-Multipliers on Weak Arens Regular Banach Algebras

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Abstract

In this paper we define the notion of weak Arens regular Banach algebras and extend the concept of quasi-multipliers to this certain class of Banach algebras. Among other the relationship between Arens regularity of the algebra A^{**} of a weak Arens regular Banach algebra A and the space $QM_r(A^*)$ of all bilinear and separately continuous right quasi-multipliers of A^* is investigated. Further, we stablish several properties of the strict and quasi-strict topologies on the $QM_r(A^*)$

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1 Introduction

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [1] for C^*-algebras. McKennon [14] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m: A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d \qquad (a, b, c, d \in A).$$

For a Banach space X, let X^* be its topological dual. The pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$. We always consider X naturally embedded into X^{**} through the mapping π, which is given by $\langle \pi(x), \xi \rangle = \langle \xi, x \rangle \quad (x \in X,\ \xi \in X^*)$.

Let A be a Banach algebra. It is well known that on the second dual A^{**} there are two algebra multiplications called the first and the second Arens product, respectively. Since in the paper we use mainly the first Arens product, we recall its definition. Let $a \in A$, $\xi \in A^*$, and $F,\ G \in A^{**}$ be arbitrary. Then one defines $\xi \cdot a$ and $G \cdot \xi$ as $\langle \xi \cdot a, b \rangle = \langle \xi, ab \rangle$ and $\langle G \cdot \xi, b \rangle = \langle G, \xi \cdot b \rangle$, where $b \in A$ is arbitrary. Now, the first Arens product of F and G is an element $F \circ G$ in A^{**} which is given by $\langle F \circ G, \xi \rangle = \langle F, G \cdot \xi \rangle$, where $\xi \in A^*$ is arbitrary. The second Arens product, which we denote by $\circ'$, is defined in a similar way.

The space A^{**} equipped with the first (or second) Arens product is a Banach algebra and A is a subalgebra of it. It is said that A is Arens regular if the equality $F \circ G = F \circ' G$ holds for all $F,\ G \in A^{**}$. For example, every C^*-algebra is Arens regular, see [3]. Note however that $F \circ a = F \circ' a$ and $a \circ F = a \circ' F$ hold for any $a \in A$ and $F \in A^{**}$.

The aim of this paper is to present a few new statements on quasi-multipliers of the dual A^* of a Banach algebra A whose second dual has a mixed identity.

In our investigation we do not assume Arens regularity, we work on certain Banach algebra which satisfies the weaker condition than Arens regularity and apply our results to C^*-algebras and to the group algebra of a compact group.

2 Weak Arens regular Banach algebras

Definition 2.1 *A Banach algebra A is called weak Arens regular if for each $\xi \in A^*$ and $F, G \in A^{**}$ we have*

$$(F \cdot \xi) \cdot G = F \cdot (\xi \cdot G)$$

Of course, every Arens regular Banach algebra is weak Arens regulsr. However, the class of weak Arens regular Banach algebras is larger. It contains, for instance, every Banach algebra A which is an ideal in its second dual.

Proposition 2.2 *Let A be a Banach algebra such that $A \trianglelefteq A^{**}$. Then A is weak Arens regular.*

Proof. Let $F,\ G \in A^{**}$ and $\xi \in A^*$ be arbitrary. We have

$$\begin{aligned}\langle (F \cdot \xi) \cdot G, a \rangle &= \langle \pi(a), (F \cdot \xi) \cdot G \rangle = \langle G \circ' \pi(a), F \cdot \xi \rangle = \langle (G \circ' \pi(a)) \circ F, \xi \rangle \\ &= \langle G \circ' (\pi(a) \circ F), \xi \rangle = \langle \pi(a) \circ F, \xi \cdot G \rangle = \langle F \cdot (\xi \cdot G), a \rangle \qquad (a \in A).\end{aligned}$$

Thus, the class of weak Arens regular Banach algebras is strictly larger than the class of Arens regular algebras.

Proposition 2.3 *Let A be a unital Banach algebra. A is weak Arens regular if and only if it is Arens regular.*

Proof. Let 1 be the identity for A, then then $\pi(1)$ is the identity for $(A^{**}, \circ)$ and $(A^{**}, \circ\prime)$. Assume that A is weak Arens regular. For arbitrary $F, G \in A^{**}$ and $\xi \in A^*$, one has

$$\begin{aligned}\langle F \circ G, \xi \rangle &= \langle F, G \cdot \xi \rangle = \langle F \circ' \pi(1), G \cdot \xi \rangle = \langle \pi(1), (G \cdot \xi) \cdot F \rangle \\ &= \langle \pi(1), G \cdot (\xi \cdot F) \rangle = \langle \pi(1) \circ G, \xi \cdot F \rangle = \langle G, \xi \cdot F \rangle = \langle F \circ' G, \xi \rangle,\end{aligned}$$

which means that weak Arens regularity implies Arens regularity.

3 Quasi-multipliers of A^* and their properties

Definition 3.1 *A bilinear mapping $m: \ A^* \times A^{**} \to A^*$ is a* right quasi-multiplier of A^* *if*

$$m(F \cdot \xi, G) = F \cdot m(\xi, G) \qquad and \qquad m(\xi, G \circ F) = m(\xi, G) \cdot F$$

hold for arbitrary $\xi \in A^*$ and $F, \ G \in A^{**}$.

Remark 3.2 *Let $QM_r(A^*)$ be the set of all bilinear and separately continuous right quasi-multipliers of A^*. It is obvious that $QM_r(A^*)$ is a linear space. Moreover, it is a Banach space with respect to the norm*

$$||m|| = \sup\{||m(\xi, F)||; \quad \xi \in A^*, \ F \in A^{**}, \ ||\xi|| \leq 1, \ ||F|| \leq 1\}.$$

of A^*.

Definition 3.3 *Let A be a general Banach algebra. Then a map $T : A^* \to A^*$ is called a right multiplier of A^* if*

$$T(F \cdot \xi) = F \cdot T(\xi),$$

for all $\xi \in A^*, F \in A^{**}$.

With $M_r(A^*)$ we denote the space of all bounded linear right multipliers on A^*. It is obvious that for each $F \in A^{**}$ the right multiplication operator $R_F\xi = \xi \cdot F$ is a right multiplier on A^*.

Theorem 3.4 *Let A be a weak Arens regular Banach algebra such that A^{**} has a mixed identity, then*

$$\rho_T(\xi, F) = (T\xi)\cdot F \qquad (T \in M_r(A^*),\ \xi \in A^*,\ F \in A^{**})$$

defines an injective linear map $\rho:\ M_r(A^*) \to QM_r(A^*)$ with norm $\|\rho\| \le 1$. Moreover, ρ is onto if A^{**} has an identity. If A^{**} has a mixed identity with norm one, then ρ is an isometry.

Proof: Let $T \in M_r(A^*)$ be arbitrary. It is obvious that ρ_T is a bilinear map from $A^* \times A^{**}$ to A^* and that it is bounded with $\|T\|$. For $a \in A$, $\xi \in A^*$, and $F,\ G \in A^{**}$, we have

$$\rho_T(F\cdot\xi, G) = T(F\cdot\xi)\cdot G = (F\cdot T\xi)\cdot G = F\cdot(T\xi\cdot G) = F\cdot\rho_T(\xi, G)$$

and

$$\rho_T(\xi, G\circ F) = (T\xi)\cdot(G\circ F) = (T\xi\cdot G)\cdot F = \rho_T(\xi, G)\cdot F.$$

Thus, $\rho_T \in QM_r(A^*)$. It follows from the definition that $\rho:\ M_r(A^*) \to QM_r(A^*)$ is linear. Obviously, $\|\rho_T\| \le \|T\|$, which gives $\|\rho\| \le 1$. Let $E \in A^{**}$ be a mixed identity. If $\rho_T = 0$, then we have $(T\xi)\cdot E = 0$ for every $\xi \in A^*$ and consequently $T = 0$. Assume that E is an identity for A^{**}. Let $m \in QM_r(A^*)$ be arbitrary. It is easily seen that $T\xi = m(\xi, E)$ $(\xi \in A^*)$ defines a bounded right multiplier of A^*. Since equalities $\rho_T(\xi, F) = (T\xi)\cdot F = m(\xi, E)\cdot F = m(\xi, E\circ F) = m(\xi, F)$ hold for all $\xi \in A^*$ and $F \in A^{**}$ we conclude that ρ is onto.

At the end assume that E is mixed identity for A^{**} of norm one. Let $T \in M_r(A^*)$ and $\varepsilon > 0$ be arbitrary. If $\xi \in A^*$ is such that $\|\xi\| \le 1$ and $\|T\| - \varepsilon < \|T\xi\|$, then

$$\|\rho_T\| \ge \|\rho_T(\xi, E)\| = \|T\xi\| > \|T\| - \varepsilon.$$

Thus, ρ is an isometry.

Remark 3.5 *Let A is a weak Arens regular Banach algebra and A^{**} has an identity. For arbitrary $m_1,\ m_2 \in QM_r(A^*)$, let $T_1,\ T_2 \in M_r(A^*)$ be uniquely determined multipliers satisfying $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$. Then*

$$m_1 \circ_\rho m_2 = \rho_{T_1} \circ_\rho \rho_{T_2} := \rho_{T_2T_1}$$

gives a well defined multiplication. It is easy to see that $QM_r(A^*)$ is a unital Banach algebra.

Define a map $\psi : A^{**} \to QM_r(A^*)$ by $\psi(H) = \rho_{R_H}$, where R_H is the right multiplication operator on A^* determined by $H \in A^{**}$. Then, for arbitrary $\xi \in A^*, F \in A^{**}$,

$$\psi(H)(\xi, F) = (\xi \cdot H) \cdot F.$$

Theorem 3.6 *Let A be a weak Arens regular Banach algebra and A^{**} be unital. Assume A^* factors on the right. Then ψ is an isomorphism of A^{**} onto $QM_r(A^*)$.*

Proof. We check only the multiplicativity of ψ since the linearity and continuity are evident. Let $H_1, H_2 \in A^{**}$. By Theorem 3.4, there exist $T_1, T_2 \in M_r(A^*)$ such that $\psi(H_1) = \rho_{T_1}$ and $\psi(H_2) = \rho_{T_2}$. Hence, for arbitrary $\xi \in A^*, F \in A^{**}$, we have

$$T_1(\xi) \cdot F = (\xi \cdot H_1) \cdot F \qquad \text{and} \qquad T_2(\xi) \cdot F = (\xi \cdot H_2) \cdot F.$$

It follows

$$\begin{aligned}(\psi(H_1) \circ_\rho \psi(H_2))(\xi, F) = \rho_{T_2 T_1}(\xi, F) = T_2(T_1(\xi)) \circ F = T_1\xi \cdot (H_2 \circ F) \\ = \xi \cdot (H_1 \circ H_2 \circ F) = \psi(H_1 \circ H_2)(\xi, F),\end{aligned}$$

which means ψ is a homomorphism.

Assume that $\psi(H) = 0$ for $H \in A^{**}$. Since the mapping ρ is one to one $R_H = 0$. Hence, for each $\xi \in A^*$, one has $\xi \circ H = 0$. Since, by the assumption, A^* factors on the right, we conclude $H = 0$. Thus, ψ is one to one. Homomorphism ψ is onto, as well. Namely, if $m \in QM_r(A^*)$, then there exist $T \in M_r(A^*)$ such that $m = \rho_T = \rho_{R_{T^*(E)}} = \psi(T^*(E))$.

Theorem 3.7 *Let A be a weak Arens regular Banach algebra and assume that A^{**} has an identity E. If A^{**} is Arens regular then $QM_r(A^*)$ is Arens regular.*

Proof: Let ψ be as in the proof of Theorem 3.6. Thus, it is an onto homomorphism. Of course, $\psi^{**} : (A^{**})^{**} \to (QM_r(A^*))^{**}$ has the same property, as well. Let $\tilde{F},\ \tilde{G} \in (QM_r(A^*))^{**}$. Then there exist $F, G \in (A^{**})^{**}$ such that $\psi^{**}(F) = \tilde{F},\ \psi^{**}(G) = \tilde{G}$. Thus,

$$\tilde{F} \circ \tilde{G} = \psi^{**}(F) \circ \psi^{**}(G) = \psi^{**}(F \circ G) = \psi^{**}(F \circ' G) = \tilde{F} \circ' \tilde{G}.$$

Remark 3.8 *Let A be a weak Arens regular Banach algebra. For each $H \in A^{**}$ and $m \in QM_r(A^*)$ we define*

$$(m * H)(f, G) = m(f, H \circ G) \qquad \textit{and} \qquad (H * m)(f, G) = m(f.H, G)$$

It is easy to see that $H * m, m * H \in QM_r(A^*)$.

4 Strict and quasi-strict topology on $QM_r(A^*)$

Beside the norm topology there are two other useful topologies on $QM_r(A^*)$. The first is the strict topology β which is given by seminorms

$$m \to \|m * F\| \qquad (F \in A^{**},\ m \in QM_r(A^*)).$$

The second is the quasi-strict topology γ. It is given by seminorms

$$m \to \|m(\xi, F)\| \qquad (\xi \in A^*,\ F \in A^{**},\ m \in QM_r(A^*)).$$

Let τ denote the topology on $QM_r(A^*)$ generated by the norm.

Theorem 4.1 *Let A be a weak Arens regular Banach algebra and A^{**} has a mixed identity. Then $(QM_r(A^*), \gamma)$, $(QM_r(A^*), \tau)$, $(QM_r(A^*), \beta)$ have the same bounded sates.*

Proof. Since $\gamma \subseteq \tau$, every $\tau-$bounded set is $\gamma-$bounded. Let H be any $\gamma-$bounded set in $QM_r(A^*)$. Then, for each $\xi \in A^*$ and $F \in A^{**}$, there exist a constant $r = r(\xi, F) > 0$ such that

$$||m(\xi, F)|| \leq r \text{ for all } m \in H. \tag{1}$$

For each $\xi \in A^*$ and $m \in H$, define $M_\xi : A^{**} \to A^*$ by

$$M_\xi(F) := m(\xi, F),\ F \in A^{**}.$$

Then $\mathcal{H} = \{M_\xi : m \in H\} \subseteq CL(A^{**}, A^*)$. By (1), for any $G \in A^{**}$,

$$||M_\xi(G)|| = ||m(\xi, G)|| \leq r(\xi, G) \ \text{ for all } \ m \in H;$$

hence $\mathcal{H}$ is pointwise bounded. Then, by the uniform boundedness principle, there exists $c = c(F) > 0$ such that

$$||M_\xi|| \leq c \text{ for all } \ m \in H. \tag{2}$$

Consider now the family $P = \{p_m : m \in H\}$ of seminorms on A^* defined by

$$p_m(\xi) = ||M_\xi|| = \sup_{||F|| \leq 1} ||M_\xi(F)|| = \sup_{||F|| \leq 1} ||m(\xi, F)||,\ \xi \in A^*.$$

For each $m \in H$, p_m is continuous on A^* since if $\{\xi_n\} \subseteq A^*$ with $\xi_n \to \xi_0$ in A^*, then

$$|p_m(\xi_n) - p_m(\xi_0)| \leq p_m(\xi_n - \xi_0) = \sup_{||F|| \leq 1} ||M_{\xi_n - \xi_0}(F)||$$

$$= \sup_{||F|| \leq 1} ||m(\xi_n - \xi_0, F)|| \to 0.$$

Then by (2), the family P is pointwise bounded. By [8, p. 142, principle 33.1] there exist a $ball B(\xi_0, r) = \{\xi \in A^* : ||\xi - \xi_0|| \leq r\}$ and a constant $k_0 > 0$ such that $p_m(\xi) \leq k_0$ for all $m \in H$ and $\xi \in B(\xi_0, r)$. For fixed $\xi \in A^*$ with $||\xi|| \leq 1$

$$p_m(\xi) = \frac{p_m(r\xi + \xi_0 - \xi_0)}{r} \leq \frac{p_m(r\xi + \xi_0) + p_m(\xi_0)}{r} \leq \frac{2k_0}{r}.$$

This implies that

$$||m|| = \sup_{||\xi|| \leq 1, ||F|| \leq 1} ||m(\xi, F)|| = \sup_{||\xi|| \leq 1} p_m(\xi) \leq \frac{2k_0}{r}.$$

(b) This follows from (a) since $\gamma \subseteq \beta \subseteq \tau$.

5 Quasi-multipliers of the dual of $L_1(G)$

At the end we consider the group algebra of a compact group G. By [19], $L_1(G)$ is Arens regular if and only if G is finite. However, since $L_1(G)$ is a two-sided ideal in its second dual ([17]), it is weak Arens regular. Note that the dual $L_1(G)^*$ can be identified with $L_\infty(G)$.

Let $M(G)$ be the convolution algebra of all bounded regular measures on G. Recall that the convolution product of $f \in L_1(G)$ and $\mu \in M(G)$ is given by

$$f * \mu(x) = \int_G f(xy^{-1})\, d\mu(y).$$

Of course, $L_\infty(G)$ is a Banach $L_1(G)^{**}$-bimodule. However, the space $L_\infty(G)$ has also a natural structure of a Banach $M(G)$-bimodule. The same holds for $L_\infty(G)^* = L_1(G)^{**}$. We will denote all these module multiplications by $*$.

Proposition 5.1 *Let G be a compact group and $A = L_1(G)$. Then the equation*

$$(\theta_\mu(\xi, F) := (\xi * \mu) * F \qquad (\mu \in M(G), \xi \in L_\infty(G),\ F \in L_1(G)^{**})$$

defines a linear isomorphism between $M(G)$ and a subspace of $QM_r(A^*)$.

Proof: Note that by the definition of module action $(\xi * \mu) * F = \xi * (\mu * F)$. From this and weak Arens regularity we conclude that $\theta_\mu \in QM_r(L_1(G)^*)$. Of course, $\theta : M(G) \to QM_r(L_1(G)^*)$ is a bounded linear map. We claim that θ is injective. Indeed, suppose that $\theta_\mu = 0$. Then $(\xi * \mu) * F = 0$ for all $\xi \in L_\infty(G)$ and $F \in (L_\infty(G))^*$. Since $L_1(G)$ has a b.a.i. it follows $\xi \circ \mu = 0$. In particular, for each $\xi \in C_0(G)$, $\xi \circ \mu = 0$. Since the measure algebra $M(G)$ is the dual of $C_0(G)$ and it has a b.a.i., $\mu = 0$, as required.

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