Compactness in Product Spaces

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Abstract

We establish some basic facts for compactness in product spaces and then derive a series of important results in analysis and measure theory.

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1 Introduction

Let $\Omega \neq \emptyset$ and $X_\omega$ be a topological space for each $\omega \in \Omega$. Let $\prod_{\omega \in \Omega} X_\omega$ be the product space with the product topology $\sigma\Omega$ which is the topology of coordinatewise convergence. For each $\omega \in \Omega$, $P_\omega : \prod_{\lambda \in \Omega} X_\lambda \to X_\omega$ is the projection such that

$$P_\omega((x_\lambda)_{\lambda \in \Omega}) = x_\omega, \quad \forall (x_\lambda)_{\lambda \in \Omega} \in \prod_{\lambda \in \Omega} X_\lambda.$$

If each $X_\omega$ is a topological vector space, then with the coordinatewise operations and the product topology $\sigma\Omega$, $\prod_{\omega \in \Omega} X_\omega$ is a topological vector space.

In this paper we would like to establish a basic proposition for compactness in the product space $\prod_{\omega \in \Omega} X_\omega$ and then we derive a series of important results in analysis and measure theory.

Observe that for $\omega \in \Omega$ and $S \subset \prod_{\omega \in \Omega} X_\omega$, $P_\omega(S) = \{x_\omega : (x_\lambda)_{\lambda \in \Omega} \in S\}$.

2 Main Results

Lemma 2.1. If $S \subset \prod_{\omega \in \Omega} X_\omega$ such that $P_\omega(S)$ is relatively compact in $X_\omega$ for each $\omega \in \Omega$, then $S$ is relatively compact in $\prod_{\omega \in \Omega} X_\omega = (\prod_{\omega \in \Omega} X_\omega, \sigma\Omega)$. 
Proof. By Tychonoff product theorem, $\prod_{\omega \in \Omega} P_\omega(S)$ is compact. If $(x_\lambda)_{\lambda \in \Omega} \in \overline{S}$, then there is a net $((x_{\alpha \lambda})_{\lambda \in \Omega})_{\alpha \in I}$ in $S$ such that $((x_{\alpha \lambda})_{\lambda \in \Omega} \xrightarrow{\sigma_\Omega} (x_\lambda)_{\lambda \in \Omega}$, i.e., $\lim_{\alpha} x_{\alpha \lambda} = x_\lambda$ for each $\lambda \in \Omega$. Hence $\overline{S} \subseteq \prod_{\omega \in \Omega} P_\omega(S) \subseteq \prod_{\omega \in \Omega} P_\omega(S)$ since $\prod_{\omega \in \Omega} P_\omega(S)$ is closed in $\prod_{\omega \in \Omega} X_\omega ([2, p.100])$. But $\prod_{\omega \in \Omega} P_\omega(S)$ is compact and $\overline{S}$ is closed in the compact $\prod_{\omega \in \Omega} P_\omega(S)$. Hence $\overline{S}$ is compact in $\prod_{\omega \in \Omega} X_\omega$. \qed

Lemma 2.2. If $X_\omega$ is Hausdorff for each $\omega \in \Omega$ and $S$ is relatively compact in $\prod_{\omega \in \Omega} X_\omega$, then $P_\omega(S)$ is relatively compact in $X_\omega$ for each $\omega \in \Omega$.

Proof. $\overline{S}$ is compact in $\prod_{\omega \in \Omega} X_\omega$ and so $P_\omega(\overline{S})$ is compact in $X_\omega$ for each $\omega \in \Omega$. Since each $X_\omega$ is Hausdorff, $P_\omega(\overline{S})$ is closed so $P_\omega(\overline{S}) = \overline{P_\omega(S)}$ for all $\omega \in \Omega$. Moreover, if $(x_\lambda)_{\lambda \in \Omega} \in \overline{S}$, i.e., there is a net $((x_{\alpha \lambda})_{\lambda \in \Omega})_{\alpha \in I}$ in $S$ such that $\lim_{\alpha} x_{\alpha \lambda} = x_\lambda$ for each $\lambda \in \Omega$, then $x_\omega = \lim_{\alpha} P_\omega((x_{\alpha \lambda})_{\lambda \in \Omega}) \in P_\omega(S)$ for each $\omega \in \Omega$ and so $(x_\omega)_{\omega \in \Omega} \in \prod_{\omega \in \Omega} P_\omega(S)$.

Hence $P_\omega(S)$ is compact in $X_\omega$ for each $\omega \in \Omega$. \qed

Corollary 2.3. If $X_\omega$ is Hausdorff for all $\omega \in \Omega$ and $S \subset \prod_{\omega \in \Omega} X_\omega$, then $S$ is relatively compact in $\prod_{\omega \in \Omega} X_\omega$ if and only if $P_\omega(S)$ is relatively compact in $X_\omega$ for each $\omega \in \Omega$.

For topological space $Y = (Y, \tau)$ and $A \subset X \subseteq Y$, let

$$A^{(X, \tau)} = \{ z \in X : \exists \text{ net } (x_\alpha)_{\alpha \in I} \text{ in } A \text{ such that } x_\alpha \xrightarrow{\tau} z \}. $$

Theorem 2.4. Let $\Omega \neq \emptyset$ and $X_\omega$ be a Hausdorff topological vector space for each $\omega \in \Omega$ and $F$ a vector subspace of $\prod_{\omega \in \Omega} X_\omega$, $S \subset F$. Let $\sigma \Omega$ be the product topology on $\prod_{\omega \in \Omega} X_\omega$. Then $S$ is relatively compact in $(F, \sigma \Omega)$ if and only if

1. $\overline{S(\prod_{\omega \in \Omega} X_\omega, \sigma \Omega)} \subset F$, i.e., $\overline{S(\prod_{\omega \in \Omega} X_\omega, \sigma \Omega)} = \overline{S(F, \sigma \Omega)}$, and

2. $P_\omega(S)$ is relatively compact in $X_\omega$ for each $\omega \in \Omega$.

Proof. Suppose that $\overline{S(F, \sigma \Omega)}$ is compact in $(F, \sigma \Omega)$. With $\sigma \Omega$ and the coordinatewise operations, $\prod_{\omega \in \Omega} X_\omega$ and its vector subspace $(F, \sigma \Omega)$ are topological vector spaces and so the compact set $\overline{S(F, \sigma \Omega)}$ is complete in $(F, \sigma \Omega)$ ([1, p.75]).

Let $((x_{\alpha \omega})_{\omega \in \Omega})_{\alpha \in I}$ be a net in $S$ such that $\lim_{\alpha} x_{\alpha \omega} \in (\prod_{\omega \in \Omega} X_\omega, \sigma \Omega)$. Since $S \subset F \subset \prod_{\omega \in \Omega} X_\omega$, the convergent net $((x_{\alpha \omega})_{\omega \in \Omega})_{\alpha \in I}$ is Cauchy in $(F, \sigma \Omega)$ and so $\lim_{\alpha} (x_{\alpha \omega})_{\omega \in \Omega} = (z_\omega)_{\omega \in \Omega} \in \overline{S(F, \sigma \Omega)} \subset F$. Since each $X_\omega$ is Hausdorff, $\prod_{\omega \in \Omega} X_\omega$ is Hausdorff ([4, p.92]) and so $((x_\omega)_{\omega \in \Omega} = (z_\omega)_{\omega \in \Omega} \in \overline{S(F, \sigma \Omega)} \subset F$. Hence $P_\omega(S)$ is relatively compact in $X_\omega$ for each $\omega \in \Omega$. \qed
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\[ \lim_{\alpha}(x_{\omega})_{\omega \in \Omega} = (z_{\omega})_{\omega \in \Omega}. \]  Hence, \((x_{\omega})_{\omega \in \Omega} = (z_{\omega})_{\omega \in \Omega} \in F\) and so \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega)) = S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega)) \subset F\). Then \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega)) = S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\) is compact in \((F, \sigma(\Omega))\) and so \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\) is compact in \((\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\). Thus, \((2)\) holds by Lemma 2.2.

Conversely, suppose that both \((1)\) and \((2)\) hold for \(S\). By Lemma 2.1, \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\) is compact in \((\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\) but \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega)) \subset F\) so \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\) is compact in \((F, \sigma(\Omega))\). \(\square\)

For a topological space \(X\) and \(\Omega \neq \emptyset\), let \(X^\Omega\) be the family of mappings from \(\Omega\) to \(X\), and letting \(X_\omega = X\) for each \(\omega \in \Omega\), the product space \((\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\) is homeomorphic with \((X^\Omega, \sigma(\Omega))\) by the correspondence \((f(\omega))_{\omega \in \Omega} \mapsto f\) for each \(f \in X^\Omega\). Then the following special case of Lemma 2.1 and Lemma 2.2 is a basic proposition in general topology ([4, p.218, Th.1]).

**Corollary 2.5.** Let \(X\) be a topological space and \(\Omega \neq \emptyset\), \(S \subset X^\Omega\). If \(S[\omega] = \{f(\omega) : f \in S\}\) is relatively compact for each \(\omega \in \Omega\), then \(S\) is relatively compact in \((X^\Omega, \sigma(\Omega))\) where \(\sigma(\Omega)\) is the topology of pointwise convergence on \(\Omega\). If, in addition, \(X\) is Hausdorff, then the converse implication also holds.

**Corollary 2.6.** Let \(X\) be a Hausdorff topological vector space, \(\Omega \neq \emptyset\) and \(F\) is a vector subspace of \(X^\Omega\), \(S \subset F\). With the topology \(\sigma(\Omega)\) of pointwise convergence on \(\Omega\), \(S\) is relatively compact in \((F, \sigma(\Omega))\) if and only if

1. \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega)) \subset F\), i.e., \(S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega)) = S(\prod_{\omega \in \Omega} X_{\omega}, \sigma(\Omega))\)

2. \(S[\omega] = \{f(\omega) : f \in S\}\) is relatively compact for each \(\omega \in \Omega\).

If \(X\) is locally convex and \(X' = \{x' \in \mathbb{C}^X : x'\text{ is linear and continuous}\}\), the dual of \(X\), then there is a beautiful duality theory for the pair \((X, X')\). However, it is possible that \(X' = \{0\}\) even for some Fréchet spaces such as \(\mathcal{M}(0, 1), L^p(0, 1)\) with \(0 < p < 1\) ([1, p.25]), etc. Hence every reasonable extension of \(X'\) will be interesting.

Let \(C(0) = \{\gamma \in \mathbb{C}^\mathbb{C} : \lim_{t \to 0} \gamma(t) = \gamma(0) = 0, |\gamma(t)| \geq |t| \text{ if } |t| \leq 1\}\). For a topological vector space \(X\), \(\gamma \in C(0)\) and \(U \in \mathcal{N}(X)\), the family of neighborhoods of \(0 \in X\), let

\[ X^{(U, \gamma)} = \{f \in \mathbb{C}^X : f(0) = 0, f \text{ is continuous, for } x \in X, u \in U \text{ and } t \in \mathbb{C}, |t| \leq 1, \]
\[ f(x + tu) = rf(x) + sf(u) \text{ where } |r - 1| \leq |\gamma(t)|, |s| \leq |\gamma(t)|\} \]

The usual dual \(X' \subset X^{(U, \gamma)}\) for all \(U \in \mathcal{N}(X)\) and \(\gamma \in C(0)\). Especially, it is possible that \(X' = \{0\}\) but \(X^{(U, \gamma)}\) is a large family ([6, 8]). Then we have a proper improvement of the Alaoglu-Bourbaki theorem as follows.

As usual, \(f_\alpha \to f\) in \((X^{(U, \gamma)}, \sigma X)\) means that \(f_\alpha(x) \to f(x)\) at each \(x \in X\), i.e., \(\sigma X\) is the weak * topology on \(X^{(U, \gamma)}\).
Theorem 2.7. Let $X$ be a topological vector space and $U \in \mathcal{N}(X)$, $\gamma \in C(0)$. If $S \subseteq X^{(U,\gamma)}$ is equicontinuous, then $S$ is relatively compact in $(X^{(U,\gamma)}, \sigma X)$, and for every $V \in \mathcal{N}(X)$, the polar $V^\circ = \{ f \in X^{(U,\gamma)} : |f(v)| \leq 1, \forall v \in V \}$ is compact in $(X^{(U,\gamma)}, \sigma X)$.

Proof. Suppose $S \subseteq X^{(U,\gamma)}$ and $S$ is equicontinuous. It is easy to see that if $x_n \to x$ in $X$ then $\lim_{n} f(x_n) = f(x)$ uniformly for $f \in S$ and so $S^{(\mathbb{C}^X, \sigma X)} \subseteq X^{(U,\sigma X)}$, $\sup_{x \in S} |f(x)| < +\infty$ at each $x \in X$, i.e., for each $x \in X$, $S[x] = \{ f(x) : f \in S \}$ is relatively compact in $\mathbb{C}$. By Corollary 2.6, $S$ is relatively compact in $(X^{(U,\gamma)}, \sigma X)$.

Let $V \in \mathcal{N}(X)$. Then $V^\circ = \{ f \in X^{(U,\gamma)} : |f(x)| \leq 1, \forall x \in V \}$ is equicontinuous on $X$ ([8, Corollary 3.6]) and $V^\circ$ is closed in $(X^{(U,\gamma)}, \sigma X)$. Thus, $V^\circ$ is compact in $(X^{(U,\gamma)}, \sigma X)$. \hfill \Box

There is a very important Vital-Hahn-Sakes-Graves-Ruess theorem ([3, 9]) says that if $X$ is a locally convex space and $S$ is a relatively compact subset in the measure space $(ca(\Sigma, X), \sigma \Sigma)$ with the topology $\sigma \Sigma$ of pointwise convergence on the $\sigma$-algebra $\Sigma$, i.e., $\mu \xrightarrow{\Sigma} \mu$ means that $\mu(A) \to \mu(A)$ at each $A \in \Sigma$, then $S$ is uniformly countably additive, i.e., if $\{ A_j \} \subseteq \Sigma$ such that $A_i \cap A_j = \emptyset (i \neq j)$, then $\lim_{n} \sum_{j=1}^{n} \mu(A_j) = \mu(\bigcup_{j=1}^{\infty} A_j)$ uniformly for $\mu \in S$. In 2006, R. Zi, Y. Yang and C. Swartz (RYC) have improved this important result to the following

RYC’s theorem. Let $X$ be a locally convex space and $\Sigma$ a $\sigma$-algebra. If $S$ is countably compact in $(ca(\Sigma, X), \sigma \Sigma)$, then $S$ is uniformly countably additive ([7, Corollary 4.3]).

Theorem 2.8. Let $X$ be a complete Hausdorff locally convex space and $\Sigma$ a $\sigma$-algebra. Then for $S \subseteq ca(\Sigma, X)$, the following (I), (II) and (III) are equivalent.

(I) $S$ is relatively compact in $(ca(\Sigma, X), \sigma \Sigma)$.

(II) $S$ is uniformly countably additive and $\{ \mu(A) : \mu \in S \}$ is compact for each $A \in \Sigma$.

(III) $S^{(X^\Sigma, \sigma \Sigma)} \subseteq ca(\Sigma, X)$ and $\{ \mu(A) : \mu \in S \}$ is compact for each $A \in \Sigma$.

Proof. (I)⇒(II). By RYC’s theorem, $S$ is uniformly countably additive. By Corollary 2.6, $\{ \mu(A) : \mu \in S \}$ is compact for each $A \in \Sigma$.

(II)⇒(III). Let $A_j \in \Sigma$, $A_i \cap A_j = \emptyset (i \neq j)$. Let $(\mu_\alpha)_{\alpha \in I}$ be a net in $S$ such that $\mu_\alpha \xrightarrow{\Sigma} \mu \in X^\Sigma$. Since $X$ is complete, $\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{\alpha} \mu_\alpha(\bigcup_{j=1}^{\infty} A_j) = \lim_{\alpha} \lim_{n} \sum_{j=1}^{n} \mu_\alpha(A_j) = \lim_{n} \lim_{\alpha} \sum_{j=1}^{n} \mu_\alpha(A_j) = \lim_{n} \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \mu(A_j)$.

Thus, $\mu \in ca(\Sigma, X)$ and so $S^{(X^\Sigma, \sigma \Sigma)} \subseteq ca(\Sigma, X)$.

(III)⇒(I). Corollary 2.6. \hfill \Box
Now the famous Bartle-Dunford-Schwartz-Nikodým theorem ([5, p.305]) can be improved to the following.

**Corollary 2.9.** Let \( \Sigma \) be a \( \sigma \)-algebra. For \( S \subset \text{ca}(\Sigma, C) \), the following (i), (ii), (iii), (iv) and (v) are equivalent.

(i) \( S \) is weakly sequentially compact ([5, p.67]).

(ii) \( S \) is uniformly countably additive and \( \sup_{\mu \in S, A \in \Sigma} |\mu(A)| < +\infty \).

(iii) \( S \) is relatively compact in \( (\text{ca}(\Sigma, C), \sigma) \).

(iv) \( S^{(C^\Sigma, \sigma)} \subset \text{ca}(\Sigma, C) \) and \( \sup_{\mu \in S} |\mu(A)| < +\infty \) for each \( A \in \Sigma \).

(v) \( S \) is uniformly countably additive and \( \sup_{\mu \in S} |\mu(A)| < +\infty \) for each \( A \in \Sigma \).

**Proof.** By Bartle-Dunford-Schwartz theorem ([5, p.305]), (i) and (ii) are equivalent.

By Theorem 2.8, (iii) is equivalent to the following.

(II') \( S \) is uniformly countably additive and \( \sup_{\mu \in S} |\mu(A)| < +\infty \) for each \( A \in \Sigma \).

By the Nikodým boundedness theorem ([5, p.309-310]), (II') is equivalent to (ii) and so (iii) \( \iff \) (ii) holds.

By Theorem 2.8, (iii) \( \iff \) (iv) \( \iff \) (v) holds.

A series of important facts in the basic theory of locally convex spaces are convenient consequences of Theorem 2.4.

**Corollary 2.10.** Banach space \( X \) is reflexive if and only if the unit ball \( B = \{ x \in X : \| x \| \leq 1 \} \) is weakly compact.

**Proof.** \( \Rightarrow \). If \( f \in B^{(C^X, \sigma X')} \), then there is a net \( (x_\alpha)_{\alpha \in I} \) in \( B \) such that \( x_\alpha \xrightarrow{\sigma X'} f \) and so \( |f(x')| = \lim_{\alpha} |x_\alpha(x')| \leq \| x' \| \) for all \( x' \in X' \), i.e., \( f \in X'' = X, \| f \| \leq 1 \), \( f \in B \). Thus, \( B^{(C^X, \sigma X')} = B \subset X \). Since \( \sup_{x \in B} |x(x')| \leq \| x' \| < +\infty \) for each \( x' \in X' \), i.e., \( B[x'] \) is relatively compact in \( C \) for each \( x' \in X' \), Theorem 2.4 shows that \( B \) is compact in \( (X, \sigma X') \).

\( \Leftarrow \). By Theorem 2.4, \( B^{(C^X, \sigma X')} \subset X \) and so \( B^{(C^X, \sigma X')} = B \). Then the Goldstine theorem shows that \( X'' = X \).

**Corollary 2.11.** A subset \( S \) of a locally convex space \( X \) is weakly compact if and only if \( S \) is bounded and complete in \( (X, \sigma X') \).
Proof. \( \Rightarrow \). \( S \subset X \subset \mathbb{C}^{X'} \). By Theorem 2.4 and the Mackey theorem ([1, p.109]), \( S \) is bounded in \( X \). Since \( S \) is compact in \( (X, \sigma X') \), \( S \) is complete in \( (X, \sigma X') \) ([1, p.75]).

\( \Leftarrow \). If \( (x_{\alpha})_{\alpha \in I} \) is a net such that \( x_{\alpha} \xrightarrow{\sigma X'} f \in \mathbb{C}^{X'} \), then \( (x_{\alpha})_{\alpha \in I} \) is Cauchy in \( (X, \sigma X') \) and so \( x_{\alpha} \xrightarrow{\sigma X'} f \in S \). Hence \( \overline{S}_{(\mathbb{C}^{X'}, \sigma X')} = S \subset X \) and \( S \) is weakly compact by Theorem 2.4.

Corollary 2.12. Let \( X \) be locally convex and \( S \subset X' \). Then \( S \) is weak * compact if and only if \( S \) is pointwise bounded on \( X \) and \( S \) is complete in \( (X', \sigma X) \).

Proof. Theorem 2.4. \( \square \)

For \( A \subset X \), \( A^o = \{ f \in X' : |f(x)| \leq 1, \forall x \in A \} \).

Corollary 2.13. Let \( X \) be a locally convex space and \( S \subset X' \). The following (a), (b) and (c) are equivalent.

(a) \( S \cap U^o \) is compact in \( (X', \sigma X) \), \( \forall U \in \mathcal{N}(X) \).

(b) \( S \cap U^o \) is closed in \( (X', \sigma X) \), \( \forall U \in \mathcal{N}(X) \).

(c) If \( (f_{\alpha})_{\alpha \in I} \) is an equiconstant net in \( S \) such that \( f_{\alpha} \xrightarrow{\sigma X} f \in \mathbb{C}^X \), then \( f \in S \).

Proof. (a) \( \Rightarrow \) (b). \( (X', \sigma X) \) is Hausdorff and \( S \cap U^o \) is compact in \( (X', \sigma X) \) for each \( U \in \mathcal{N}(X) \) and so (b) holds.

(b) \( \Rightarrow \) (c). \( [(f_{\alpha})_{\alpha \in I}]^o \in \mathcal{N}(X) \), \( (f_{\alpha})_{\alpha \in I} \subset S \cap [(f_{\alpha})_{\alpha \in I}]^o \) and \( f_{\alpha} \xrightarrow{\sigma X} f \). Then \( f \in S \cap [(f_{\alpha})_{\alpha \in I}]^o \) and so \( f \in S \).

(c) \( \Rightarrow \) (a). Let \( U \in \mathcal{N}(X) \). \( U \subset U^{oo} \), \( U^{oo} \in \mathcal{N}(X) \) and hence \( U^{oo} \) is equiconstant. Let \( (f_{\alpha})_{\alpha \in I} \) be a net in \( S \cap U^o \) such that \( f_{\alpha} \xrightarrow{\sigma X} f \in \mathbb{C}^X \). By (c), \( f \in S \). But \( |f(u)| = \lim_{\alpha} |f_{\alpha}(u)| \leq 1 \) for all \( u \in U \) and so \( f \in X' \), \( f \in S \cap U^o \), \( S \cap U^{oo} = S \cap U^o \subset X' \). If \( x \in X \) and \( f \in S \cap U^o \), then \( \frac{1}{n_0} x \in U \) for some \( n_0 \in \mathbb{N} \) and so \( |f(\frac{1}{n_0} x)| \leq 1, |f(x)| \leq n_0 \), \( \{f(x) : f \in S \cap U^o\} \) is compact for each \( x \in X \). Thus, (a) follows from Theorem 2.4. \( \square \)

A topological vector space is BTB if each bounded set in \( X \) is totally bounded ([1, p.85]), e.g., \( \mathbb{R}^n \), \( \mathbb{C}^n \) and the space \( D \) of text functions, etc. If \( X \) is complete BTB, then bounded sets in \( X \) are relatively compact. Theorem 2.4 can be used to show that we have BTB spaces as many as all sets.

Corollary 2.14. If \( X \) is a complete BTB space, then for every \( \Omega \neq \emptyset \) the product space \( (X^{\Omega}, \sigma \Omega) \) is a BTB space.
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**Proof.** Let \( S \) be a bounded set in \((X^\Omega, \sigma_\Omega)\). Then \( \{g(\omega) : g \in S\} \) is totally bounded in \( X \) for each \( \omega \in \Omega \) and so \( \{g(\omega) : g \in S\} \) is compact in \( X \) for each \( \omega \in \Omega \). By Theorem 2.4 or Corollary 2.5, \( S \) is relatively compact in \((X^\Omega, \sigma_\Omega)\). Thus, \((X^\Omega, \sigma_\Omega)\) is BTB. \( \square \)

**Corollary 2.15.** Let \( X \) be a topological vector space and \( Y \) a complete Hausdorff BTB space, \( S \subset L(X,Y) = \{f \in Y^X : f \, \text{is linear and continuous}\} \).
If \( S \) is equicontinuous, then \( S(\mathcal{Y}^X, \sigma X) \) is compact in \((L(X,Y), \sigma X)\).

**Proof.** Since \( S \) is equicontinuous, \( S(\mathcal{Y}^X, \sigma X) \subset L(X,Y) \), i.e., \( S(\mathcal{Y}^X, \sigma X) = S'(L(X,Y), \sigma X) \).
By the equicontinuity of \( S \) again, \( S \) is pointwise bounded but the range space \( Y \) is complete BTB and so \( \{g(x) : g \in S\} \) is compact for each \( x \in X \). Then the desired follows from Theorem 2.4. \( \square \)

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**References**


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