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# The Fekete Szegö Problem for a Subclass of Quasi-Convex Functions of Order $\beta$ Type $\gamma$ with Respect to Symmetric Points

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**Abstract**

In [2], Janteng et al. introduced new subclass of $S$ denoted by $K_s^*(\alpha, \beta, \gamma)$ for $0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$. This paper obtained sharp upper bounds for $|a_n|, n = 2, 3$ and the Fekete-szegö inequalities for functions $f \in K_s^*(\alpha, \beta, \gamma)$.

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## 1 Introduction

Let $S$ be the class of functions $f$ which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

where $a_n$ is a complex number.

For $0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$, Janteng et al. in [2] were introduced a new class of functions denoted by $K_s^*(\alpha, \beta, \gamma)$.

**Definition 1.1** *Let $f$ be given by (1). Then for $0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$, $f \in K_s^*(\alpha, \beta, \gamma)$ if there exists a $g \in C_s(\gamma)$ such that for $z \in D$,*

$$Re\left\{\frac{2\alpha(z^2f''(z))'}{(g(z)-g(-z))'} + \frac{2(zf'(z))'}{(g(z)-g(-z))'}\right\} > \beta.$$

Note: The definition is also equivalent to the following:
$f \in K_s^*(\alpha, \beta, \gamma)$ if there exists a $h = zg'(z) \in S_s^*(\gamma)$ such that

$$Re\left\{\frac{2\alpha z(z^2f''(z))'}{h(z)-h(-z)} + \frac{2z(zf'(z))'}{h(z)-h(-z)}\right\} > \beta. \tag{2}$$

We note that the class $K_s^*(\alpha, 0, 0) = K_s^*(\alpha)$ in [1].

# 2 Preliminary Results

Here is a preliminary lemma required for proving our results.

**Lemma 2.1** *([3]) Let $k$ be analytic in $D$ with $Re\{k(z)\} > 0$ and be given by $k(z) = 1 + c_1z + c_2z^2 + \ldots$ for $z \in D$, then*

$$|c_n| \leq 2 (n \geq 1)$$

*and*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

# 3 Main Result

**Theorem 3.1** *Let $f \in K_s^*(\alpha, \beta, \gamma), 0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$ and be given by by (1), then*

$$2(\alpha + 1)|a_2| \leq 1 - \beta$$

*and*

$$9(2\alpha + 1)|a_3| \leq 3 - 2\beta - \gamma$$

*Proof.*
Since $h \in S_s^*(\gamma)$, it follows that

$$2zh'(z) = [\gamma + (1-\gamma)H(z)](h(z) - h(-z))$$

for $z \in D$, with $Re\{H(z)\} > 0$ where $H(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$ Upon equating coefficients, we obtain

$$2b_2 = (1-\gamma)p_1, 2b_3 = (1-\gamma)p_2 \tag{3}$$

It follows from (2) that

$$2\alpha z(z^2 f''(z))' + 2z(zf'(z))' = [\beta + (1-\beta)k(z)](h(z) - h(-z)) \tag{4}$$

where $Re\{k(z)\} > 0$. Writing $k(z) = 1 + c_1 z + c_2 z^2 + ...$ and equating coefficients in (4) gives

$$4(\alpha+1)a_2 = (1-\beta)c_1, 9(2\alpha+1)a_3 = (1-\beta)c_2 + b_3 \tag{5}$$

The result now follows on using classical inequalities $|p_n| \leq 2, |c_n| \leq 2, n \geq 2$ and the inequality $|b_3| \leq 1-\gamma$ which follow from (3).

Now we consider the functional $|a_3 - \mu a_2^2|$ for a complex $\mu$.

**Theorem 3.2** *For $f \in K_s^*(\alpha, \beta, \gamma), 0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$ and $\mu$ complex,*

$$|a_3 - \mu a_2^2| \leq \frac{3-2\beta-\gamma}{9(2\alpha+1)} max \left( 1, \frac{4(1-\gamma)(\alpha+1)^2 + (1-\beta)|8(\alpha+1)^2 - 9\mu(2\alpha+1)(1-\beta)|}{4(3-2\beta-\gamma)(\alpha+1)^2} \right)$$

*The result obtained is sharp.*

*Proof.*
From (5), we write

$$a_3 - \mu a_2^2 = \frac{1-\beta}{9(2\alpha+1)}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(1-\beta)\left\{8(\alpha+1)^2 - 9\mu(2\alpha+1)(1-\beta)\right\}c_1^2}{144(2\alpha+1)(\alpha+1)^2} + \frac{(1-\gamma)p_2}{18(2\alpha+1)} \tag{6}$$

It follows from (6) and Lemma 2.1 that

$$|a_3-\mu a_2^2| \leq \frac{1-\beta}{9(2\alpha+1)}\left|c_2-\frac{c_1^2}{2}\right| + \frac{(1-\beta)|8(\alpha+1)^2-9\mu(2\alpha+1)(1-\beta)||c_1|^2}{144(2\alpha+1)(\alpha+1)^2}$$

$$+\frac{(1-\gamma)|p_2|}{18(2\alpha+1)} \tag{7}$$

$$\leq \frac{1-\beta}{9(2\alpha+1)}\left\{2-\frac{|c_1|^2}{2}\right\} + \frac{(1-\beta)|8(\alpha+1)^2-9\mu(2\alpha+1)(1-\beta)||c_1|^2}{144(2\alpha+1)(\alpha+1)^2}$$

$$+\frac{(1-\gamma)|p_2|}{18(2\alpha+1)}$$

$$= \frac{2(1-\beta)}{9(2\alpha+1)} + \frac{(1-\beta)\left\{|8(\alpha+1)^2-9\mu(2\alpha+1)(1-\beta)|-8(\alpha+1)^2\right\}|c_1|^2}{144(2\alpha+1)(\alpha+1)^2}$$

$$+\frac{(1-\gamma)|p_2|}{18(2\alpha+1)} \tag{8}$$

which, on using $|c_1| \leq 2$ and $|p_2| \leq 2$ gives

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{3-2\beta-\gamma}{9(2\alpha+1)}, & \text{if } \kappa(\alpha,\beta) \leq 8(\alpha+1)^2; \\ \frac{1-\gamma}{9(2\alpha+1)} + \frac{(1-\beta)|8(\alpha+1)^2-9\mu(2\alpha+1)(1-\beta)|}{36(2\alpha+1)(\alpha+1)^2}, & \text{if } \kappa(\alpha,\beta) \geq 8(\alpha+1)^2. \end{cases}$$

where $\kappa(\alpha,\beta) = |8(\alpha+1)^2-9\mu(2\alpha+1)(1-\beta)|$.

Letting $c_1 = 0, c_2 = p_2 = 2$ and $c_1 = c_2 = p_2 = 2$ respectively in (6) shows that the result is sharp.

Next, we consider the real number $\mu$ as follows.

**Theorem 3.3** *For $f \in K_s^*(\alpha,\beta,\gamma), 0 \leq \alpha < 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$ and $\mu$ real,*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{3-2\beta-\gamma}{9(2\alpha+1)} - \frac{\mu(1-\beta)^2}{4(\alpha+1)^2}, & \text{if } \mu \leq 0; \\ \frac{3-2\beta-\gamma}{9(2\alpha+1)}, & \text{if } 0 \leq \mu \leq \frac{16(\alpha+1)^2}{9(2\alpha+1)(1-\beta)}; \\ \frac{\mu(1-\beta)^2}{4(\alpha+1)^2} - \frac{1-2\beta+\gamma}{9(2\alpha+1)}, & \text{if } \mu \geq \frac{16(\alpha+1)^2}{9(2\alpha+1)(1-\beta)}. \end{cases}$$

*The result obtained is sharp.*

*Proof.*
We consider two cases. At first, we suppose that $\mu \leq \frac{8(\alpha+1)^2}{9(2\alpha+1)(1-\beta)}$. From (8) and using the fact that $|c_1| \leq 2, |p_2| \leq 2$, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{3-2\beta-\gamma}{9(2\alpha+1)} - \frac{\mu(1-\beta)^2}{4(\alpha+1)^2}, & \text{if } \mu \leq 0; \\ \frac{3-2\beta-\gamma}{9(2\alpha+1)}, & \text{if } 0 \leq \mu \leq \frac{8(\alpha+1)^2}{9(2\alpha+1)(1-\beta)}. \end{cases}$$

Letting $c_1 = c_2 = p_2 = 2$ and $c_1 = 0, c_2 = p_2 = 2$ respectively in (6) shows that the result is sharp.

Next, we suppose that $\mu \geq \frac{8(\alpha+1)^2}{9(2\alpha+1)(1-\beta)}$. In this case, it follows from (8) and Lemma 2.1 that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{3-2\beta-\gamma}{9(2\alpha+1)}, & \text{if } \frac{8(\alpha+1)^2}{9(2\alpha+1)(1-\beta)} \leq \mu \leq \frac{16(\alpha+1)^2}{9(2\alpha+1)(1-\beta)}; \\ \frac{\mu(1-\beta)^2}{4(\alpha+1)^2} - \frac{1-2\beta+\gamma}{9(2\alpha+1)}, & \text{if } \mu \geq \frac{16(\alpha+1)^2}{9(2\alpha+1)(1-\beta)}. \end{cases}$$

Letting $c_1 = 0, c_2 = p_2 = 2$ and $c_1 = 2i, c_2 = -2, p_2 = 2$ respectively in (6) shows that the result is sharp.
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