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On Some Dynamical Properties

of Unimodal Maps

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Abstract

A Unimodal map is a continuous function from [0,1] into itself, for which, and has unique critical point. In this work, we study the some dynamical properties of Unimodal maps and we shows that, if f with negative Schwarzian derivative then f is chaotic map, and we study multi types of chaos and find relation between define on it.

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1-Introduction

A Dynamical systems given by the iteration of a continuous map on an interval were broadly studied because they are simple and nevertheless they exhibit complex behaviors. Moreover they allow numerical simulations using a

computer, which enabled to discover some chaotic phenomena. However continuous interval maps have many properties that are not found on other spaces.
As a consequence , the study of one dimensional dynamical is very rich but not necessarily representative of other systems. Iteration of continuous maps of an interval into it self serve as the simplest examples of models for dynamical systems.

An interval map $f:[0,1]\to[0,1]$ is called Unimodal map if $f(0)=f(1)=0$ and has only one critical point [8]. Unimodal maps are interesting for both physical and mathematical reasons. They have the capacity to generate extremely complicated behavior in spite of their apparent simplicity. They have been used to modal everything from insect populations to the on set of turbulence with varying degrees of success[8].

Their mathematical study goes back to the early years of last century when Fatou and Julia [3] already knew that some Unimodal maps have infinitely many periodic points. Further development was slow until the seventies, since when the field has greatly expanded and matured.

The topic of this paper is the study of iterates of f , we shall denote f^0 = identity , $f^1 = f$, $f^2 = fof$, $f^n = fof^{n-1}, n>2$.
The study of the iterates f is interesting if we consider the orbits $O_f(x)=\{x, f(x), f^2(x),...\}$ of $x\in[0,1]$ .A point x is called periodic for f if $O_f(x)$ is a finite set ($f^n(x)=x,$ for some $n>0$), and $O_f(x)$ is called the periodic orbit of x . A periodic point x of period n then x is a fixed point of f^n . We recalled x is an attracting (repelling)periodic point if x is an attracting (repelling) fixed point of f^n (a fixed point is called attracting (repelling)fixed if $|f'(x)|<1$ ($|f'(x)|>1$))[3].

If x is attracting (repelling) then each point in $\{x, f(x), f^2(x),..., f^{n-1}(x)\}$ is attracting (repelling) periodic point. So we say that cycle attracting (repelling)[3].

In the 1960s as well ,Sharkovskii began to study the structure of systems given by a continuous map on an interval, in particular the co-existence of periodic points of various periods , which is ruled by Sharkovskii's order[1]. In the paper " period three implies chaos"[13]Li and Yorke proved that a continuous interval map with a periodic point of period 3 has periodic points of all periods ,which is actually a part of Sharkovskii's Theorem. They also proved that for such a map f there exists an uncountable set of points such that ,if x, y are two distinct point in this set then $f^n(x), f^n(y)$ are arbitrarily closed for some n and are father than some positive distance for other integers n tending to infinity ; the term "chaos" was introduced in mathematics through the paper of Li and Yorke where it was used to name this behavior[13].

After wards various definitions of chaos were proposed . the map $f : X \to X$, (where X is the space) is said sensitive to initial condition if near every point x there exist arbitrarily close points y such that the distance between $f^n(x), f^n(y)$ is greater than a given $\delta > 0$ for some n . The chaos in the sense of Li-Yorke asks for more instability but only on a subset[3].

For Devaney[4] ,chaos is see as mixing of unpredictability and regular behaviors: a system is chaotic in the sense of Devaney if it is transitive ,sensitive to initial conditions and has a dense set of periodic points.

Our first aim is to study and classification of Unimodal map. The second aim is to study multi types of chaos on Unimodal map and find the relation between them.

2- Some Property of Unimodal Map

In this section we recall the definition of Unimodal maps, and simply introduce the particular forms and notations used in this paper[4].
A Unimodal map is a continuous function $f : [0,1] \to [0,1]$ for which :

(1) $f(0) = f(1) = 0$
(2)The exists c in $(0,1)$ such that f is strictly increasing on $[0, c]$ and strictly decreasing on $[c,1]$.

Recall that ,if f is a differentiable function on an interval $[0,1]$, then c is a critical point in interior of $[0,1]$ if $f'(c)=0$. The Unimodal map has a unique critical point. For example ,the quadratic map $f_\mu(x)=\mu x(1-x)$ is Unimodal map with critical point $c=1/2$ where μ is in the range $0<\mu\le 4$.
Let f be a C^3 map , i.e., it has three continuous derivatives . The Schwarzian derivative at a point x is given by[6], $S(f)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}(\frac{f''(x)}{f'(x)})^2$, and denoted by $S(f)$

This derivative was first formulated by H.A. Schwarzian and has been used in the theory of differential equation. It has found important application in study of bifurcation of application in the study of bifurcation of periodic orbits, the Schwarzian derivative is used to study the limiting behavior of dynamical systems [6]. If f has a negative Schwarzian derivative on $[0,1]$, then it turns out that there must be a number c in $(0,1)$ such that $f'(c)=0$,that is , f has a critical point. We say that f is S-Unimodal if it Schwarzian derivative is negative[8],[3] .for example every function of the form $f(x)=1-\mu x^2$, with $0<\mu\le 2$ is S-Unimodal map.

On the other hand ,maps with negative Schwarzian derivative have many special properties which do not hold for other maps. For example , an S-Unimodal map can have at most two attracting periodic points while an arbitrary Unimodal map can have arbitrarily many attracting periodic points.
In [5], prove the following result.

Lemma(2-1):
let f have a finite number of critical points, and assume that $S(f)<0$.Then for any positive integer m there is finite number of period m points of f.

We are prepared to state Singers theorem .For convenience, when we refer to cycles, we will include fixed points[5].

Theorem(2-2) (Singers Theorem):

Let f be defined on closed interval J, and suppose that $f(J) \subset J$ .Assume that $S(f) < 0$,and f has n critical points. Then f has at most $n+2$ attracting cycles.

Now , we can generalizing Singers theorem for all S-Unimodal map.

Corollary (2-3):

Every S-Unimodal has at most three attracting cycles.

Proof :

let $f : [0,1] \to [0,1]$ be S-Unimodal map with $S(f) < 0$,since f has unique critical point ,then by Singers theorem , f has at most 1+2 attracting cycles. □

Let f be Unimodal map ,and let $O_f(x)$ be a periodic orbit of period p. We call $O_f(x)$ a stable periodic orbit if for $y \in O_f(x)$, $|Df^p(y)| \le 1$. By the chain rule of differentiation , $Df^p(y)$ takes the same value for all $y \in O_f(x)$. So that the definition make sense. The importance of stable periodic orbits for dynamical system comes from the following observation. If $O_f(x)$ is stable periodic orbit for f ,(of period p) ,then there is some neighborhood U of $y \in O_f(x)$ such that $\lim_{n\to\infty} f^{np}(z) = y$ for some $z \in U$ except possibly if $|Df^p(y)| = 1$ [3].

The main result in[3] is:

Theorem(2-4):

If f is S-Unimodal, then every stable periodic orbit attracts at least one of the end points of interval ,or the critical point.
From the above result ,it easy prove the following corollary:

Corollary(2-5):

There exist S-Unimodal function without a stable periodic orbit.

Singer D. in [9] ,prove the following result .

Theorem (2-6):
Let $f:[0,1]\rightarrow[0,1]$ be a C^3 map and let it satisfy :
(1) $f(0)=f(1)=0$
(2) f has a unique critical point c in $(0,1)$.
(3) $S(f)<0$ for all $x\in[0,1]-c$ .Then f has at most one stable orbit in $(0,1)$.

Now we can classification the S-Unimodal in two class first without stable periodic orbit and the other with stable periodic orbit.

Theorem(2-7):
Every S-Unimodal either has stable periodic orbit or without stable periodic orbit.

Proof:
Let f be S-Unimodal map ,then by corollary(2-3) , f has at most three attracting cycles . that mean it has at most three orbits with $|Df^n(x)|\leq 1$ for some n . Thus f has at most three stable orbit.
From above ,the S-Unimodal map can be with stable periodic orbit ,then by corollary (2-5) ,there exist S-Unimodal map without stable periodic orbit. □

The another class of Unimodal map is called C -class of S-Unimodal map if[8] :

- Non-flat critical point : there exist $l>1$ and $L>1$ such that $|x-c|^{l-1}/L \leq |f'(x)| \leq L|x-c|^{l-1}$,for all $x\in[0,1]$.
- $|f'(0)|>1$.

for example the quadratic map $q_\mu(x)=\mu x(1-x)$ is Unimodal map(with critical point c=1/2) , $0<\mu\leq 4$, belongs to the class C of S-Unimodal map.

The tent map is composed of two straight line segment of slope λ

$$T_\lambda(x) = \begin{cases} \lambda x & 0 \le x \le 1/2 \\ \lambda(1-x) & 1/2 \le x \le 1 \end{cases}$$

It is Unimodal map if λ lies in (0,2]. But no tent map belongs to the class C, we will recall some of the basic properties of T_λ [7]: T_λ is transitive ,has a dense set of periodic points and the topological entropy $h(T_\lambda) = \log \lambda$.

Adler ,Konheim and McAndrew ,introduced the topological entropy in 1965 as an invariant of continuous maps. A measure of complexity ,it describes the rate at which observing an orbit imperfectly gives information about its initial point. They defined the topological entropy for any continuous mapping from a compact topological space to itself. For Unimodal maps there is an equivalent definition due to Misiurewicz and Szlenk [3]: $h(f) = \lim_{n\to\infty} \frac{1}{n} \log N_n$ Where N_n is the number of laps(i.e. monotone pieces) of f^n.

This limit always exists. The topological entropy of any Unimodal map therefore lies in the range[0,log2]. Every value in this range is taken on by some Unimodal map. Milnor and Thurston, prove that any Unimodal map is semi conjugate to tent map T_λ [8]

Theorem(2-8)[8]:

If f is a Unimodal map with positive topological entropy then it is semi conjugate to T_λ with $h(f) = \log \lambda$.

For the above result, we have the topological entropy of Unimodal map is non-negative i.e. $h(f) = 0$ or $h(f) > 0$.

3- Chaos on Unimodal Maps

Chaotic behavior is a manifestation of the complexity of nonlinear dynamical systems. There are several different definition of chaos, which describe the complexity of dynamical system in different a spaces. A well know definition of chaos is given by Li and Yorke[13],whose main characteristic is the existence of uncountable scrambled set .Another famous definition of chaos is given by Devaney [4],whose characteristic is sensitive dependence on initial conditions with transitive and density of periodic points. Sensitive dependence is widely understood as the central idea in Devaney's definition of chaos , but it is implied by transitivity and density of periodic points see[7].

A continuous map $f : I \to I$,where I is the unit interval ,is chaos of Li& Yorke if there is an Uncountable set $S \subset I$ such that orbit of any two distinct point x, y in S are :
$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.

In [13] , shows that period 3 implies chaos of Li &Yorke.

Theorem(3-1):
Let f be a Unimodal map with a periodic point of period 3.Then there is an uncountable set S of points and $\varepsilon > 0$ such that for every $x, y \in S, x \neq y$, $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \varepsilon$.

Now, we can prove the following corollary .

Corollary(3-2):
Any S-Unimodal map is Li& Yorke chaos.

Proof: Let f be S-Unimodal map then by Lemma(2-1) , f has periodic orbit of period m ,for any $m > 0$, let $m = 3$,by Theorem(3-1), f is Li &Yorke chaos. □

Now ,let f be Unimodal map without stable periodic orbit, we discuss a notion of sensitive dependence on initial condition. We say f has sensitivity on initial condition at x if there is an $\varepsilon > 0$ such that for each $\delta > 0$,there is a y in I and positive integer n such that $|x - y| < \delta$ and $|f^n(x) - f^n(y)| > \varepsilon$ [6].

We first know that a map with stable periodic orbit does not have sensitivity .

Theorem(3-3)[3]:

If f is S-Unimodal and has a stable periodic orbit of period k then f has no sensitivity.

The second type of chaos is a widely studied dynamical phenomena in topological dynamics. The most used definition of chaos was due to Devaney. In Devaney's definition of chaos[4] , a dynamical system is chaotic if it satisfies the following three condition ,namely (a) f is topological transitive(for every pair of non empty open sets U,V in I ,there is a positive integer n such that $f^n(U) \cap V \neq \phi$.)

(b) periodic points of f are dense (c) f is sensitive of initial conditions.

However ,in later developments ,it was shown that (a) and (b) imply (c) , in some spaces Thus topological transitivity is necessary condition for exhibit chaos. For an interval map, it is even worse :the transitivity are enough to imply the other two conditions ,as it was pointed out by Silverman [10].

Let $f : X \to X$ and $g : Y \to Y$ be continuous maps, recall that f is semi-conjugate with the map g, if there exists a map $h : X \to Y$ satisfying $h \circ f = g \circ h$ and h is onto. Murad Banaji in [2].

Lemma(3-4) [2]:

If f and g are semi-conjugate ,if f is topologically transitive , then so is g .

Now ,we can prove the following result :

Theorem(3-5):

Every Unimodal map with positive topological entropy is Devaney chaos.

Proof:

Let f be Unimodal map with positive topological entropy, thus by theorem(2-8), f is semi conjugate to T_λ. Since T_λ is transitive map and by lemma(3-4), f is transitive map thus f is Devaney chaos. □

Since ,if f is Devaney chaos then f has positive topological entropy ,and from theorem(3-5) we can say : if f is Unimodal map the Devaney chaos and positive topological entropy are equivalent .

The third type of chaos is ω-chaos ,briefly ωc, introduced in 1993 by Shihai Li in [11]. Let $f : I \to I$ be a continuous map. A subset S of I is called an $\omega - scrambled$ set for the map f if , for any $x, y \in S$ with $x \neq y$, the following conditions hold :

(i) $\omega(x, f) \setminus \omega(y, f)$ is uncountable.

(ii) $\omega(x, f) \cap \omega(y, f) \neq \phi$.

(iii) $\omega(x, f) \not\subset P(f)$.

where $\omega(x, f)$ is the ω-limit set of a point $x \in I$ and $P(f)$ is set of periodic points. f is called ω-chaos if there exists an uncountable $\omega - scrambled$. In [11], prove that in interval map ω-chaos and positive topological entropy are equivalent .

Theorem(3-6):

If f be continuous map of a compact interval to itself .then ω-chaos and positive topological entropy is equivalent.

Let f be continuous map of a compact interval to itself , for any pair (x, y) of points in I and any positive integer $i \in N$, the distance of the iterations denote by $\delta_{xy}(i) = d(f^i(x), f^i(y))$, where d is a metric on the interval. For real t , and any positive integer n define a distribution function[12]: $\varsigma(x,y,t,n): R \to [0,1]$ by

$$\varsigma(x,y,t,n) = \frac{1}{n}\#\{i : 0 \le i < n \, and \quad \delta_{xy}(i)\}$$

Obviously, $\varsigma(x,y,t,n)$ is non-decreasing function, $\varsigma(x,y,t,n) = 0$ for $t \le 0$ and $\varsigma(x,y,t,n) = 1$

for t greater than diameter of I .Put $F^{*}_{xy}(t) = \lim_{n\to\infty}\sup \xi(x,y,t,n)$

$$F_{xy}(t) = \lim_{n\to\infty}\inf \zeta(x,y,t,n)$$

Then $F_{xy}(t)$ is called the lower distribution function, and $F^{*}_{xy}(t)$ the upper distribution function of x and y .Obviously, $0 \le F_{xy}(t) \le F^{*}_{xy}(t) \le 1$ for any real t and $F^{*}_{xy}(t) = 0$ for $t < 0$, $F_{xy}(t) = 1$ for $t > 1$.The map . If $F_{xy}(t) < F^{*}_{xy}(t)$ for all $t \in [0,1]$, we simply write $F_{xy} < F^{*}_{xy}$. The map f is distributional chaos if there is a set $D \subset I$ containing at least two points such that for any $x \ne y$ in D, $F_{xy} < F^{*}_{xy}$ for all t in an interval .

It is known that ,for continuous map of the unit interval, the distribution chaos and positive topological entropy are equivalent [12] .

If Unimodal map with positive topological entropy then f is distribution chaos. Now, from the above result we can prove the following theory

Theorem(3-7):

Let f be S-Unimodal map then the following statement are equivalent;

1- f is positive topological entropy.

2- f is Devaney chaos.

3- f is Li& Yorke chaos.

4- f is ω-chaos.

5- f is distribution chaos.

Proof: Directly from the above result. □

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