COMPENSATION PAYMENTS FOR AIRCRAFT NOISE IN AN URBAN BUILDING CONFLICT SITUATION ON THE BASIS OF A SET-VALUED CONJUGATE DUALITY

Norman Neukel

HIKARI LTD

HIKARI LTD

Hikari Ltd is a publisher of international scientific journals and books.

www.m-hikari.com

Norman Neukel, *Compensation Payments for Aircraft Noise in an Urban Building Conflict Situation on the Basis of a Set-Valued Conjugate Duality*

Norman Neukel
Leithe 5 96120 Bischberg-Bamberg, Germany
neukel@t-online.de

ISBN 978-954-91999-8-7 (print)
ISBN 978-954-91999-9-4 (online)

Typeset using MS Word.

Mathematics Subject Classification: 90B50, 90B70, 90B90, 91D10, 91D20

Keywords: Set-valued conjugate duality, partial order, socio-economy

Published by Hikari Ltd

PREFACE

This book investigates compensation payments to property owners for aircraft noise in urban conflict situations in the region surrounding the airport for the first time on the basis of a set-valued conjugate duality. This optimal perturbation approach serves as justification for the realization of variable results. This "dual" socio-economy indicates the action strategy of different interest groups. Linear point sets as a new optimum set varying according to socio-economic properties and a payment function are therefore well substantiated.

Norman Neukel
02 January, 2018

Contents

1. Introduction

In a vector optimization problem all optimum points of a non-empty subset set must be found for given target functions. In addition to the question of the existence and characterization of minimal elements enable the representation of the perturbed target function by the embedding of the primary problem in a family of so-called perturbation problems, a dual problem in the form of a conjugate map. In this conjugate duality theory every solution of the stable primary problem is allocated to a specific solution of the dual problem, which is characterized as a subgradient of the perturbed allowed value map.

The motivation for the conjugate dual problem can be found in the definition. With a perturbation function one influences the mapping and the argument of the target function. What relationship exists between the optimum target function of the perturbed dual problem and the primary problem? A weak duality theorem states that the elements of the set-valued target map images and the elements of the related negative conjugates of a perturbation map fulfill a certain inequality.

This duality is based on conjugate vector-valued functions introduced in 1949 by Fenchel [6] and refined by Rockafellar [21] in regard to scalar optimality problems. The conjugate duality approach, based on the allowed elements in finite-dimensional spaces, was described by Sawaragi, Nakayama and Tanino [22] and Tanino and Sawaragi [28]. The approach was further elaborated by Lalitha and Arora [15].

Other publications which followed this approach stem from Brumelle [5], Kawasaki [13] and Luc [16]. Kawasaki extends the set notation of a supremum in regard to weak minimality and maximality to a closed set. Similar results can be found in Tanino [26] and [27] in Euclidean vector spaces and partially ordered topological vector spaces. Song [25] extended the results above to a general case and presented related stability criteria. Subdifferentials are investigated in the research work of Hernández and Rodríguez-Marín [9] on the basis of Azimov and Gasimov [2], [3] and Kasimbeyli and Mammadov [12]. Azimov and Gasimov [3] describe conditions for non-convex vector optimization problems with the application of an extended Lagrange function in the weak sense for non-convex functions in an ordered vector space with auxiliary cones for the epigraph of a perturbation function. In [14] Y. Kücük, Atasever and M. Kücük introduce general weak subdifferentials. Schrage [24] describes convex conjugate and biconjugate set-valued maps of the Fenchel-Rockafellar type. This gives the duality results, which are closely related Hamel [7]. On the basis of the subdifferential of a convex set-valued function the difference between the classical vector-values scalarization and the set-value analysis is described. Hamel and Schrage [8] discuss a theoretical and algebraic framework for the extended real numbers with the terms $+\infty$ and $-\infty$ as a "residual set".

Neukel [20] investigates airport emission influences on urban conflict situations with unordered as-built properties and their effects on the local real estate market using set-valued order relations. In an example for the determination of publicly registered land values Neukel [17] utilizes interval-valued duality and regression, resulting in an optimum, describing the linking of an interval-valued and a set-valued approach. With the help of a dual equivalence class model extended by an order relation, Neukel [19] explains a new decision strategy for the example of the determination of an optimum set of publicly registered and property ownership prices in the Federal Republic of Germany. This results in an adjusted data / point set.

In this book the conjugates, biconjugates and the subdifferential of a set-valued map are presented on the basis of the approach Sawaragi, Nakayama and Tanino [22], Tanino and Sawaragi [28] and Bot, Grad and Wanka [4]. Here the theory in [22] and [28] is extended to the infinitely dimensioned vector space, contrary to [4] however in order to avoid certain difficulties (see Schiel [23]) this is not based on the elements "$+\infty$" and "$-\infty$". Optimal and minimal elements are treated in topologically infinite vector spaces. In this section, these maps are represented by a partial order – similar to certainly less – in relation to a convex cone in finite vector spaces. The properties of these notations are described analogously to those of vector-valued origin. A general perturbation approach for a set-valued dual problem is then defined, with which compensation payments to property owners in Frankfurt am Main for aircraft noise are investigated for the first time. The application of set-valued optimization with reference to a duality yields a varying optimum point set as the new result in regard to freely chosen as-built properties and payment functions. This action strategy utilizes a partial order. The resulting conclusion shows: with the application of this conjugate duality model in an urban conflict situation that the results can be derived abductively from a single case.

This book is structured as follows: Section 2 describes the conjugate duality model for the set optimization. The concept of conjugate mapping achieves a duality and the solution of both a primal and a dual problem with the concept of conjugate mapping. In Section 3 this duality model is allocated socio-economically to the urban building conflict situation (the urban region surrounding the Frankfurt airport), resulting in new decision processes for compensation payments to property owners based on a variable optimum.

2. Conjugate duality in set optimization: Solution concept, conjugate mapping and subdifferentials

Let Y be an arbitrary real linear vector space and $C_Y \subset Y$ a convex cone. This cone induces a preorder $\leq_Y$ on Y:

For $a,\ b \in Y:\ b - a \in C_Y \ \Leftrightarrow:\ a \leq_Y b$. (1.1)

In the following we address restricted set optimization problems. Let X, Y be real vector spaces, S a non-empty subset of X, $C_Y \subset Y$ a convex and $F : X \rightrightarrows Y$ a set-valued map with

$$S \subset dom(F) := \{x \in X \mid F(x) \neq \emptyset\}.$$

We then examine the minimization problem:

$$\leq_Y - \min_{x \in S} F(x). \qquad (P\ 1)$$

Definition 2.1

A pair (x, y) with $x \in S$ and $y \in F(x)$ is referred to as the *minimizer* of $(P\ 1)$ if

$$F(x) \cap \operatorname{Min}(F(S), \leq_Y) \neq \emptyset, \text{ where } F(S) = \bigcup_{x \in S} F(x).$$

Note that a minimizer is an element of the graph of F.
For maximization problems a maximizer is introduced analogously.

Every point $x \in S$ is *allowed* with regard to $(P\ 1)$ and S also describes the *admissible range* and the *set of all admissible points* of $(P\ 1)$.

In this book, this vector access, i.e. the investigation of the problem $(P\ 1)$ with the optimality concept of the minimizer, is treated in a new socio-economic context.

Initially, it is necessary to introduce preparatory notations with regard to the minimality and maximality in topological vector spaces. In preparation for the duality model, in this section the definitions and related results of the vector optimization are therefore represented within a general framework.

In this section let X, Y be the topological vector spaces and X^*, Y^* the related topological dual spaces. Furthermore, let Y be partially ordered by a non-trivial, peaked and convex cone $C_Y \subset Y$. C_{Y^*} describes the *topological dual cone* of C_Y in Y^*. The definitions of the minimal and maximal elements of a general non-empty subset relative to a partial order $\leq_Y$ are given (see Definition 2.1 in Neukel [19]).

With the above convex cone $C_Y \subset Y$ a partial order $\leq_Y$ is induced on *Y*. Since C_Y is also peaked, $\leq_Y$ is antisymmetric (see Definition 2.1 in Neukel [19]). For $x \leq_Y y$ the analogous notation $y \geq_Y x$ is used. $x \not\leq_Y y$ implies that $y - x \notin C_Y$. Furthermore, here $x <_Y y$ if $y - x \in C_Y$ and $x \neq y$ (see Definition 2.1, minimizer). For the characterization of the conjugate duality which follows $\leq_Y$ is used.

As a consequence of property (d), page 9 in Neukel [18] the structure of the certainly less order relation $\preccurlyeq_c$ (see Neukel [20], Definition 2.3) is equivalent to the buildup of $\leq_Y$.

The following properties relating to the $\operatorname{Min}(\mathcal{A}, \leq_Y)$ and $\operatorname{Max}(\mathcal{A}, \leq_Y)$ for $\emptyset \neq \mathcal{A} \subset Y$ (see Neukel [18]) apply:
$\operatorname{Max}(\mathcal{A}, \leq_Y) = \operatorname{Min}_{-C_Y}(\mathcal{A}, \leq_Y) = -\operatorname{Min}(-\mathcal{A}, \leq_Y)$, where $\operatorname{Min}_{-C_Y}(\mathcal{A}, \leq_Y)$ describes the set of all minimal elements of $\mathcal{A}$, induced on $-C_Y$.

With regard to the solution concept of the minimizer, assume $F : X \rightrightarrows Y$ to be a set-valued map with the graphs $graph(F)$, with the original image region $dom(F)$ given. For a non-empty subset $S \subset X$ we describe $F(S) := \bigcup_{x \in S} F(x)$.

$\mathcal{C}(X, Y)$ describes the set of all continuously linear mappings of *X* to *Y*.

The following definition derives from the definition of the conjugates, biconjugates, subgradients and subdifferential of a function with values from the real vector space (see also Sawaragi, Nakayama and Tanino [22]).

This section examines the perturbation concept for set-valued optimization problems (see Sawaragi, Nakayama and Tanino [22] and Bot, Grad and Wanka [4]).
Here $F : X \rightrightarrows Y$ is a set-valued map, where $dom(F)$ is a non-empty set.

By analogy to $(P\ 1)$ the general set-valued optimization problem is:

$$\leq_Y - \min_{x \in X} F(x). \qquad (P\ 2)$$

Here it is necessary to determine the minimizers (see Definition 2.1). The vector access is utilized.

Definition 2.2

Let $F : X \rightrightarrows Y$ be given as a set-valued map with $F(x) \neq \varnothing$ for all $x \in X$.

(a) The set-valued map

$$F^* : \mathcal{C}(X,Y) \rightrightarrows Y,\ F^*(T) := \mathrm{Max}\left(\bigcup_{x \in X} \{T(x)\} - F(x), \leq_Y \right),\ T \in \mathcal{C}(X,Y)$$

represents the *conjugate map* of *F* if $F^*(T) \neq \varnothing$.

(b) The set-valued map

$$F^{**} : X \rightrightarrows Y,\ F^{**}(x) := \mathrm{Max}\left(\bigcup_{T \in \mathcal{C}(X,Y)} \{T(x)\} - F^*(T), \leq_Y \right)$$

represents the *biconjugate map* of *F if* $F^*(T) \neq \varnothing$.

(c) The operator $T \in \mathcal{C}(X,Y)$ represents a subgradient of *F* in $(x,y) \in graph(F)$, if $T(x) - y \in \mathrm{Max}\left(\bigcup_{y \in X} \{T(y)\} - F(y), \leq_Y \right)$.

The set of all subgradients of *F* in $(x,y) \in graph(F)$ represents the *subdifferential* of *F* in (x,y) and takes the form $\partial F(x,y)$. Furthermore, for all $x \in X$: $\partial F(x) := \bigcup_{y \in F(x)} \partial F(x,y)$. If for all $y \in F(x)$ the relation $\partial F(x,y) \neq \varnothing$ is valid, then *F* is subdifferentiable in *x*.

Notations exist for the concepts of supremum and infimum corresponding to the previously described maximum and minimum. However, these have numerical disadvantages. This treatment of the supremum on a set closure derives from Ansari, Yang and Yao [1]. The conjugate duality can also be represented by "weak" concepts (weak minimality and maximality). This approach allows the construction of a theory based on the notations of the weak type (see Bot, Grad and Wanka [4]).

The treatment of the vector-valued functions $f : X \to Y$ follows analogously to Definition 2.2. For all $T \in \mathcal{C}(X,Y)$ the following formula applies for the conjugate of *f*:

$$f^*(T) = \mathrm{Max}\left(\bigcup_{x \in X} T(x) - f(x), \leq_Y \right) \text{ for } x \in X \text{ with } f^*(T) \neq \varnothing.$$

For $T \in \mathcal{C}(X,Y)$ the subgradient of *f* at the point $(\bar{x}, f(\bar{x}))$ is with $\bar{x} \in X$, then *T* is a s*ubgradient* of *f* in $\bar{x}$ in the usual sense.

Accordingly, $T(\bar{x}) - f(\bar{x}) \in \mathrm{Max}\left(\bigcup_{x\in X} Tx - f(x), \leq_Y\right)$, equivalent to the condition that no $x \in X$ exists with $T(x) - f(x) >_Y T(\bar{x}) - f(\bar{x})$. In the result: For all $x \in X$ $f(x) - f(\bar{x}) \not<_Y T(x - \bar{x})$. By definition, the order relation $<_Y$ excludes $x \neq \bar{x}$.

The set of all subgradients of *f* in $\bar{x} \in X$ represents the *subdifferential* of *f* at the point $\bar{x}$ and takes the form $\partial f(\bar{x})$. For example, for $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $f : X \to \mathbb{R}$, $\bar{x} \in X$ and $T \in \mathcal{C}(X, \mathbb{R}) = X^*$:

$$f(x) - f(\bar{x}) \not<_Y T(x - \bar{x}) \;\; \forall x \in X \;\; \Leftrightarrow \;\; f(x) - f(\bar{x}) \geq T(x - \bar{x}) \;\; \forall x \in X.$$

This result is the classical definition of the scalar subgradient (see Definition 2.1.23 in Sawaragi, Nakayama and Tanino [22]).

Example 2.3:

Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued map with $F(x) = \{y \,|\, |x| \leq y\}$. The conjugate map of *F* for every $T \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ with $T(x) = \alpha x,\ \alpha \in \mathbb{R}$:is then:

$$F^*(T) = \mathrm{Max}\left(\bigcup_{x\in\mathbb{R}} \{\alpha x\} - \{y \,|\, |x| \leq y\}, \leq\right) = \begin{Bmatrix} \{0\} \text{ for } |\alpha| \leq 1 \\ \varnothing \text{ for } |\alpha| > 1 \end{Bmatrix}.$$

Justification: Since $|x|$ is bounded by *y* and $F(x)$ is closed maximal elements exist for the case $|\alpha| \leq 1$. For $|\alpha| > 1$ none exist. The related biconjugate map is

$$\begin{aligned} F^{**}(x) &= \mathrm{Max}\left(\bigcup_{\alpha\in\mathbb{R}} \{\alpha x\} - \{0\}, \leq\right) \\ &= \mathrm{Max}\left(\bigcup_{|\alpha|\leq 1} \{\alpha x\} - \{0\}, \leq\right) \\ &= \mathrm{Max}\left(\bigcup_{|\alpha|\leq 1} \{\alpha x\}, \leq\right) \\ &= \begin{Bmatrix} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{Bmatrix} \\ &= |x| \text{ for all } x \in \mathbb{R}. \end{aligned}$$

The conjugate map of the set-valued mapping $F : X \rightrightarrows Y$ with $F(x) \neq \varnothing$ for all $x \in X$ possesses useful properties. For this we define $G : X \rightrightarrows Y$, $G(x) \neq \varnothing$ for all $x \in X$, with $G(x) = F(x - \bar{x})$ with an arbitrarily chosen $\bar{x} \in X$. It then follows from the generalization of Propositions 6.1.1 and 6.1.2 in Sawaragi, Nakayama and Tanino [22]:

(a) $G^*(T) = F^*(T) + T(\bar{x})$ for all $T \in \mathcal{L}(X,Y)$ with $G^*(T) \neq \emptyset$ and $F^*(T) \neq \emptyset$,

(b) $G^{**}(x) = F^{**}(x - \bar{x})$ for all $x \in X$ with $G^{**}(x) \neq \emptyset$ and $F^{**}(x - \bar{x}) \neq \emptyset$,

(c) $(F + \bar{y})^*(T) = F^*(T) - \{\bar{y}\}$ for $\bar{y} \in Y$ with $(F + \bar{y})^*(T) \neq \emptyset$ and $F^*(T) \neq \emptyset$,

(d) $(F + \bar{y})^{**}(x) = F^{**}(x) + \{\bar{y}\}$ for $\bar{y} \in Y$ with $(F + \bar{y})^{**}(x) \neq \emptyset$ and $F^{**}(x) \neq \emptyset$.

(c) and (d) are the conjugate representation "set + point".

Proof:

(a) For $T \in \mathcal{L}(X,Y)$

$$G^*(T) = \mathrm{Max}\left(\bigcup_{x \in X}\{T(x)\} - G(x), \leq_Y\right) = \mathrm{Max}\left(\bigcup_{x \in X}\{T(x)\} - F(x - \bar{x}), \leq_Y\right)$$

$$= \mathrm{Max}\left(\bigcup_{y \in X}\{T(y + \bar{x})\} - F(y), \leq_Y\right) = \mathrm{Max}\left(\bigcup_{y \in X}\{T(y)\} - F(y), \leq_Y\right) + \{T(\bar{x})\}$$

$$= F^*(T) + \{T(\bar{x})\}.$$

(b) For $x \in X$

$$G^{**}(x) = \mathrm{Max}\left(\bigcup_{T \in \mathcal{L}(X,Y)}\{T(x)\} - G^*(T), \leq_Y\right)$$

$$= \mathrm{Max}\left(\bigcup_{T \in \mathcal{L}(X,Y)}\{T(x - \bar{x})\} - F^*(T), \leq_Y\right) = F^{**}(x - \bar{x}).$$

(c) For $\bar{y} \in Y$

$$(F + \bar{y})^*(T) = \mathrm{Max}\left(\bigcup_{x \in X}\{T(x)\} - (F + \bar{y})(x), \leq_Y\right)$$

$$= \mathrm{Max}\left(\bigcup_{x \in X}\{T(x)\} - F(x), \leq_Y\right) - \{\bar{y}\} = F^*(T) - \{\bar{y}\}.$$

(d) For $\bar{y} \in Y$:

$$(F + \bar{y})^{**}(x) = \mathrm{Max}\left(\bigcup_{T \in \mathcal{C}(X,Y)}\{T(x)\} - (F + \bar{y})^*(T), \leq_Y\right)$$

$$= \mathrm{Max}\left(\bigcup_{T \in \mathcal{C}(X,Y)}\{T(x)\} - F^*(T), \leq_Y\right) + \{\bar{y}\} = F^{**}(x) + \{\bar{y}\}.$$

Proposition 2.4 (Generalization of the Young-Fenchel inequality)
Let $F : X \rightrightarrows Y$ be a set-valued map and $T \in \mathcal{C}(X,Y)$. For all $y \in F(x)$ with $x \in X$ and all $y^* \in F^*(T)$ it then follows that

$$y + y^* \not\leq_Y T(x). \tag{2.1}$$

Proof:
Let $x \in X$ and $T \in \mathcal{L}(X,Y)$ be chosen as fixed and let $y \in F(x)$ and

$y^* \in F^*(T) = \mathrm{Max}\left(\bigcup_{x \in X} \{T(x)\} - F(x), \leq_Y \right).$

With reference to the definition of the $\leq_Y$ maximality of y^* (no value exists that is greater than y^*), then $y^* \not\leq_Y T(x) - y$, proving (2.1). □

Proposition 2.4 can also be found in Bot, Grad and Wanka [4] for an extended vector space *Y*. However, in the present book in order to avoid certain difficulties we do not utilize the elements $"+\infty"$ and $"-\infty"$ (see also Schiel [23]).

In the scalar case the inequality $f^{**} \leq f$ is always true. An analogous result for set-valued maps can be found in the following proposition (see Corollary 6.1.1 in Sawaragi, Nakayama and Tanino [22]).

In the following, the elementary properties of conjugate maps and subgradients are given (see Proposition 6.1.5 (ii) and Proposition 6.1.6 (ii) in Sawaragi, Nakayama and Tanino [22]).

Proposition 2.5 is a simple remark.

Proposition 2.5
Let $F : X \rightrightarrows Y$ be a set-valued map. Then for

$(x, y) \in graph(F)$ with $F(x) \neq \emptyset$ and $F^*(T) \neq \emptyset$:

$$T \in \partial F(x, y) \Leftrightarrow T(x) - y \in F^*(T).$$

Proof:
This proposition is a direct consequence of Definition 2.2, (c). □

Proposition 2.6
Let $F : X \rightrightarrows Y$ be a set-valued map and $(x, y) \in graph(F)$. Then with $F^{**}(x) \neq \emptyset$

$$\partial F(x, y) \neq \emptyset \Leftrightarrow y \in F^{**}(x).$$

It then follows that we can write:

$$F \text{ is subdifferentiable at the point } x \iff F(x) \subset F^{**}(x).$$

Proof:

Let $(x, y) \in graph(F)$.

(a) Let $\partial F(x, y) \neq \emptyset$. For an arbitrarily chosen $\bar{T} \in \partial F(x, y)$, according to Proposition 2.5

$$\bar{T}(x) - y \in F^*(\bar{T}) \text{ or}$$

$$y \in \{\bar{T}(x)\} - F^*(\bar{T}) \subset \bigcup_{T \in \mathcal{C}(X,Y)} \{T(x)\} - F^*(T).$$

Proposition 2.4 states that $y \not<_Y T(x) - y^*$ for all $T \in \mathcal{L}(X,Y)$ and all $y^* \in F^*(T)$.

But this means that no $\tilde{y} \in \bigcup_{T \in \mathcal{C}(X,Y)} \{T(x)\} - F^*(T)$ exists with $\tilde{y} <_Y y$, and therefore $y \in \operatorname{Max}\left(\bigcup_{T \in \mathcal{C}(X,Y)} \{T(x)\} - F^*(T), \leq_Y \right) = F^{**}(x)$.

(b) Now let $y \in F^{**}(x)$. By definition it follows that $y \in \{\bar{T}(x)\} - F^*(\bar{T})$ for a $\bar{T} \in \mathcal{L}(X,Y)$. Proposition 2.5 leads to $\bar{T} \in \partial F(x, y)$.

The first equivalence is thus proven.

The equivalent characterization of the subdifferentiability of F in x follows directly from Definition 2.2, (c) and the previously proven equivalence. □

Lemma 2.7

Let $\mathcal{A}_1, \mathcal{A}_2 \subset Y$ with the given sets $\mathcal{A}_1, \mathcal{A}_2 \neq \emptyset$.

(a) If $\operatorname{Min}(\mathcal{A}_1, \leq_Y) \neq \emptyset$ and $\operatorname{Min}(\mathcal{A}_2, \leq_Y) \neq \emptyset$, then

$$\operatorname{Min}(\mathcal{A}_1 + \mathcal{A}_2, \leq_Y) \subset \operatorname{Min}(\mathcal{A}_1, \leq_Y) + \operatorname{Min}(\mathcal{A}_2, \leq_Y).$$

(b) If $\operatorname{Max}(\mathcal{A}_1, \leq_Y) \neq \emptyset$ and $\operatorname{Max}(\mathcal{A}_2, \leq_Y) \neq \emptyset$, then

$$\operatorname{Max}(\mathcal{A}_1 + \mathcal{A}_2, \leq_Y) \subset \operatorname{Max}(\mathcal{A}_1, \leq_Y) + \operatorname{Max}(\mathcal{A}_2, \leq_Y).$$

Proof:

(a) Let $y \in \operatorname{Min}(\mathcal{A}_1 + \mathcal{A}_2, \leq_Y)$ be arbitrarily chosen.

A $y_1 \in \mathcal{A}_1$ and a $y_2 \in \mathcal{A}_2$ exist so that $y = y_1 + y_2$. If $y_1 \notin \operatorname{Min}(\mathcal{A}_1, \leq_Y)$, there is a $\bar{y}_1 \in \mathcal{A}_1$, so that $\bar{y}_1 <_Y y_1$. It then follows that $\bar{y}_1 + y_2 \in \mathcal{A}_1 + \mathcal{A}_2$ and $\bar{y}_1 + y_2 <_Y y_1 + y_2 = y$, contradicting the assertion that y is a minimal element of $\mathcal{A}_1 + \mathcal{A}_2$. Then $y_1 \in \operatorname{Min}(\mathcal{A}_1, \leq_Y)$. Analogously $y_2 \in \operatorname{Min}(\mathcal{A}_2, \leq_Y)$ can be proven.

(b) This part follows from (a), taking into account $\mathrm{Max}\left(\mathcal{A},\leq_Y\right)=-\mathrm{Min}\left(-\mathcal{A},\leq_Y\right)$ for $\mathcal{A}\subset Y$. □

The inclusions from Lemma 2.7 are valid without additional assumptions.

Assuming $\mathrm{Min}\left(\mathcal{A}_1,\leq_Y\right)=\varnothing$ and $\mathrm{Min}\left(\mathcal{A}_2,\leq_Y\right)=\varnothing$, then

$\mathrm{Min}\left(\mathcal{A}_1+\mathcal{A}_2,\leq_Y\right)=\varnothing$ (see the proof of Lemma 2.7, (a)).

Definition 2.8
Let $\mathcal{A}\subset Y$ with $\mathcal{A}\neq\varnothing$ be a given set.
(a) The set $\mathrm{Min}\left(\mathcal{A},\leq_Y\right)$ is *externally stable* if

$$\mathcal{A}\subset\mathrm{Min}\left(\mathcal{A},\leq_Y\right)+C_Y.$$

(b) The set $\mathrm{Max}\left(\mathcal{A},\leq_Y\right)$ is *externally stable* if

$$\mathcal{A}\subset\mathrm{Max}\left(\mathcal{A},\leq_Y\right)-C_Y.$$

The external stability plays an important role in the considerations that follow. The properties above are referred to in the technical literature as the *domination* property. This was introduced for the first time by Vogel [29].

Definition 2.9
Let C_Y be a convex cone in $Y:=\mathbb{R}^n$. $cl\left(C_Y\right)$ describes the *closure* of C_Y. A non-empty set $\mathcal{A}\subset\mathbb{R}^n$ is C_Y*-compact* if the set $\left(\{y\}-cl\left(C_Y\right)\right)\cap\mathcal{A}$ is compact for every $y\in\mathcal{A}$.

If C_Y is a peaked, convex and closed cone in $\mathbb{R}^n$ and $\mathcal{A}\subset\mathbb{R}^n$ a non-empty, C_Y-compact set, then external stability exists for $\mathrm{Min}\left(\mathcal{A},\leq_Y\right)$, i.e. $\mathcal{A}\subset\mathrm{Min}\left(\mathcal{A},\leq_Y\right)+C_Y$ (see Sawaragi, Nakayama and Tanino [22], Theorem 3.2.9).

The following Lemma 2.10 extends Lemma 6.1.1 in Sawaragi, Nakayama and Tanino [22]:

Lemma 2.10
Let $F,G:X\rightrightarrows Y$ be set-valued maps with $F\left(x\right)\neq\varnothing$, $G\left(x\right)\neq\varnothing$ and $\mathrm{Max}\left(G\left(x\right),\leq_Y\right)\neq\varnothing$ for all $x\in X$. Then

$$\left.\begin{aligned} &\operatorname{Max}\left(\bigcup_{x\in X} F(x)+G(x),\leq_Y\right)\\ &\subset \operatorname{Max}\left(\bigcup_{x\in X} F(x)+\operatorname{Max}\left(G(x),\leq_Y\right),\leq_Y\right).\end{aligned}\right\} \tag{2.2}$$

If $\operatorname{Max}\left(G(x),\leq_Y\right)$ is also externally stable for all $x\in X$, then the equality of (2.2) is satisfied.

Proof:

(a) The existence of the inclusion must be shown (2.2). For this we choose an arbitrary

$y\in \operatorname{Max}\left(\bigcup_{x\in X} F(x)+G(x),\leq_Y\right)$.

An $\overline{x}\in X$, $y_1\in F(\overline{x})$ and $y_2\in G(\overline{x})$, so that $y=y_1+y_2$. If $y_2\notin \operatorname{Max}\left(G(\overline{x}),\leq_Y\right)$, then by definition there is a $\overline{y}_2\in G(\overline{x})$ with $y_2<_Y \overline{y}_2$. It then follows that $y=y_1+y_2<_Y y_1+\overline{y}_2 \in \bigcup_{x\in X} F(x)+G(x)$ and this contradicts the maximality of *y*. Therefore $y_2\in \operatorname{Max}\left(G(x),\leq_Y\right)$ and furthermore $y=y_1+y_2\in \bigcup_{x\in X} F(x)+\operatorname{Max}\left(G(x),\leq_Y\right)$. That *y* is a maximal element of the set $\bigcup_{x\in X} F(x)+\operatorname{Max}\left(G(x),\leq_Y\right)$ results from

$\bigcup_{x\in X} F(x)+\operatorname{Max}\left(G(x),\leq_Y\right)\subset \bigcup_{x\in X} F(x)+G(x)$.

Assuming (b) $\operatorname{Max}\left(G(x),\leq_Y\right)$ is externally stable for $x\in X$, then

$G(x)-C_Y=\operatorname{Max}\left(G(x),\leq_Y\right)-C_Y$ for all $x\in X$, because

$G(x)-C_Y\subset \operatorname{Max}\left(G(x),\leq_Y\right)-C_Y$ and the equality follows because of $\operatorname{Max}\left(G(x),\leq_Y\right)\subset G(x)$. Then:

$F(x)+G(x)-C_Y=F(x)+\operatorname{Max}\left(G(x),\leq_Y\right)-C_Y$ for all $x\in X$ and

$\bigcup_{x\in X}\left(F(x)+G(x)\right)-C_Y=\bigcup_{x\in X}\left(F(x)+\operatorname{Max}\left(G(x),\leq_Y\right)\right)-C_Y$.

Proposition 3.1.2 from Sawaragi, Nakayama and Tanino [22], a standard statement that is also valid in infinite dimensionality, then yields

$\operatorname{Max}\left(\bigcup_{x\in X}\left(F(x)+G(x)\right),\leq_Y\right)=\operatorname{Max}\left(\bigcup_{x\in X}\left(F(x)+\operatorname{Max}\left(G(x),\leq_Y\right)\right),\leq_Y\right)$,

justifying the assertion. □

The following Corollary 2.11 extends Corollary 6.1.1 in Sawaragi, Nakayama and Tanino [22]:

Corollary 2.11
Let $F: X \rightrightarrows Y$, $F(x) \neq \emptyset$ be a set-valued map for every $x \in X$ and $\operatorname{Max}(F(x), \leq_Y) \neq \emptyset$. For every $T \in \mathcal{C}(X,Y)$ it then follows that

$$F^*(T) \subset \operatorname{Max}\left(\bigcup_{x \in X} \{T(x)\} - \operatorname{Min}(F(x), \leq_Y), \leq_Y\right).$$

If $\operatorname{Min}(F(x), \leq_Y)$ is externally stable for all $x \in X$, then the equality is also satisfied.

Proof:
Let $T \in \mathcal{C}(X,Y)$ be arbitrarily chosen.
We then examine the maps $\tilde{F}: X \rightrightarrows Y$, $\tilde{F}(x) = \{T(x)\}$ for all $x \in X$ and $\tilde{G}: X \rightrightarrows Y$, $\tilde{G}(x) = -F(x)$ for all $x \in X$. Since

$$\operatorname{Max}(\tilde{G}(x), \leq_Y) = \operatorname{Max}(-F(x), \leq_Y) = -\operatorname{Min}(F(x), \leq_Y),$$

the external stability of $\operatorname{Min}(F(x), \leq_Y)$ guarantees the external stability of $\operatorname{Max}(\tilde{G}(x), \leq_Y)$. Replacing the given F by $\tilde{F}$ and the given G by $\tilde{G}$ in Lemma 2.10 leads to the desired inclusions. The proof of equality is equivalent to Lemma 2.10. □

Corollary 2.12
Let $F: X \rightrightarrows Y$ with $F(x) \neq \emptyset$ for all $x \in X$ be a set-valued map and $\emptyset \neq \operatorname{Max}(F(x), \leq_Y)$ be externally stable for all $x \in X$. Then

$$\operatorname{Max}\left(\bigcup_{x \in X} F(x), \leq_Y\right) = \operatorname{Max}\left(\bigcup_{x \in X} \operatorname{Max}(F(x), \leq_Y), \leq_Y\right).$$

Proof:
This makes use of Lemma 2.10. Here we choose the mapping of G as F (in this corollary), and the map F in Lemma 2.10 is chosen so that all image sets are equal to $\{0_Y\}$. □

The propositions of Lemma 2.10, Corollary 2.11 and Corollary 2.12 can also be formulated for minimal elements. For example, with suitable assumptions:

$$\operatorname{Min}\left(\bigcup_{x \in X} F(x), \leq_Y\right) = \operatorname{Min}\left(\bigcup_{x \in X} \operatorname{Min}(F(x), \leq_Y), \leq_Y\right)$$

if $\operatorname{Min}\left(\bigcup_{x \in X} F(x), \leq_Y\right)$ is externally stable for all $x \in X$.

In Lemma 2.10, Corollary 2.11 and Corollary 2.12 the external stability cannot be omitted. An example of this can be found in Remark 6.1.1 of Sawaragi, Nakayama and Tanino [22]):

Let $F:\mathbb{R}\rightrightarrows\mathbb{R}^2$ with

$$F(x)=\begin{Bmatrix}\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, \text{ if } x=0\\ \left\{(y_1,y_2)\in\mathbb{R}^2 \,\middle|\, y_1^2+y_2^2<3\right\}, \text{ if } x\neq 0\end{Bmatrix} \text{ for all } x\in\mathbb{R}.$$

$F(x)$ represents an open circular area for $x\neq 0$ that contains no maximal.

Then for $T=(0,0)^T$: $F^*(T)=\varnothing$,

$$\operatorname{Max}\left(\bigcup_{x\in\mathbb{R}}\{T(x)\}-\operatorname{Min}\left(F(x),\leq\right),\leq\right)=\{0\}.$$

Analogously, from the definition of $F(x)$ for $x=0$ and $x\neq 0$:

$$\operatorname{Max}\left(\bigcup_{x\in\mathbb{R}}F(x),\leq\right)=\varnothing, \text{ but } \operatorname{Max}\left(\bigcup_{x\in\mathbb{R}}\operatorname{Max}\left(F(x),\leq\right),\leq\right)=\{0\}.$$

The assertion in Corollary 2.12 is therefore not valid.

For a closed circular area the example above does not function.

In the following the perturbation concept for set-valued optimization problems (see Sawaragi, Nakayama and Tanino [22]) is examined. Let $F:X\rightrightarrows Y$ be a set-valued map, where $dom(F)$ is a non-empty set. Let us examine the $(P\ 2)$ again from the point of view of the concept of the minimizer.

A set-valued dual problem relating to $(P\ 2)$ is then constructed and a *set-valued perturbation map* $\Phi: X\times W\rightrightarrows Y$ introduced, so that $\Phi(x,0_W)=F(x)$ for all $x\in X$. Here W describes a topological vector space and is referred to as the *perturbation space* and W' decribes its dual space (the designation of the map Φ derives from Sawaragi, Nakayama, and Tanino [22]).

$(P\ 2)$ is therefore embedded in a family of perturbation problems:

$$\leq_Y -\min_{x\in X}\Phi(x,w). \qquad (P\ 3)$$

$w\in W$ describes the *perturbation variable*. $(P\ 3)$ with $w=0_W$ then agrees with $(P\ 2)$.

The dual problem can be represented with the help of the *conjugate of the perturbation map*:

$$\Phi^* : \mathcal{C}(X,Y)\times\mathcal{C}(W,Y) \rightrightarrows Y,$$

$$\Phi^*(T,\Lambda) := \operatorname{Max}\left(\bigcup_{x\in X,\, w\in W}\{T(x)+\Lambda(w)\}-\Phi(x,w),\leq_Y\right),$$

if $\operatorname{Max}\left(\bigcup_{x\in X,\, w\in W}\{T(x)+\Lambda(w)\}-\Phi(x,w),\leq_Y\right)\neq\varnothing$.

The primal problem $(P\ 2)$ can be allocated to the set-valued dual problem as a dual function:

$$\leq_Y - \max_{\substack{\text{with the constraints}\\ \Lambda\in\mathcal{C}(W,Y)}} -\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right). \qquad (D\ 1)$$

The dual variables of this problem are operators.

A $\overline{\Lambda}\in\mathcal{C}(W,Y)$ is required so that a $\overline{y}^*\in-\Phi^*\left(0_{\mathcal{C}(X,Y)},\overline{\Lambda}\right)$ exists with $\overline{y}^*\in\operatorname{Max}\left(\bigcup_{\Lambda\in\mathcal{C}(W,Y)}-\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right),\leq_Y\right)$. In this case $\overline{\Lambda}\in\mathcal{C}(W,Y)$ is *admissible* with regard to $(D\ 1)$ and $\left(\overline{\Lambda},\overline{y}^*\right)$ a maximizer of $(D\ 1)$ (see Definition 2.1).

The weak duality is valid within this framework.

In the following theorem the previously elaborated definition of Φ^* excludes the empty set:

Theorem 2.13 (weak duality for (*P* 2) and (*D* 1))
For all $x\in X$ and all $\Lambda\in\mathcal{C}(W,Y)$ with $\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)\subset F^*\left(0_{\mathcal{C}(X,Y)}\right)$

$$\Phi(x,0_W)\cap\left\{-\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)-C_Y\setminus\{0_Y\}\right\}=\varnothing.$$

Proof:
Let $x\in X$ and $\Lambda\in\mathcal{C}(W,Y)$ be arbitrarily chosen. Assume that a $y\in\Phi(x,0_W)\cap\left\{-\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)-C_Y\setminus\{0_Y\}\right\}$ exists, i.e. it is in fact $y\in F(x)\cap\left\{-\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)-C_Y\setminus\{0_Y\}\right\}$. Then a $y^*\in\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)$ exists so that $y+y^*\in-C_Y\setminus\{0_Y\}$. According to our assumption $y^*\in F^*\left(0_{\mathcal{C}(X,Y)}\right)$ and then,

following Proposition 2.4, $y+y^* \not<_Y 0_Y$, so that $y+y^* \notin -C_Y \setminus \{0_Y\}$ contradicts $y+y^* \in -C_Y \setminus \{0_Y\}$. □

Corollary 2.14
For all $x \in X$ and all $\Lambda \in \mathcal{C}(W,Y)$

$$y \not<_Y y^* \text{ for all } y \in F(x) \text{ and all } y^* \in -\Phi^*\left(0_{\mathcal{C}(X,Y)}, \Lambda\right).$$

Proof:
This proposition with an order relation is evidently an equivalent formulation of Theorem 2.13, utilizing an intersecting set. □

The following corollary offers a direct conclusion in relation to the solutions for $(P\ 2)$ and $(D\ 1)$:

Corollary 2.15
Let $\bar{y} \in F(\bar{x}) \cap \left\{-\Phi^*\left(0_{\mathcal{C}(X,Y)}, \bar{\Lambda}\right)\right\}$ be given for an $\bar{x} \in X$ and $\bar{\Lambda} \in \mathcal{C}(W,Y)$. Then $(\bar{x}, \bar{y})$ is a minimzer of $(P\ 2)$, while $(\bar{\Lambda}, \bar{y})$ is a maximizer of $(D\ 1)$.

Proof:
Assuming that $(\bar{x}, \bar{y})$ is not a minimzer of $(P\ 2)$, then an $x \in X$ and $y \in F(x) = \Phi(x, 0_W)$ exist so that $y <_Y \bar{y}$. It then follows that $y \in \Phi(x, 0_W) \cap \left\{-\Phi^*\left(0_{\mathcal{C}(X,Y)}, \Lambda\right) - C_Y \setminus \{0_Y\}\right\}$, since $\bar{y} \in F(\bar{x}) \cap \left\{-\Phi^*\left(0_{\mathcal{C}(X,Y)}, \bar{\Lambda}\right)\right\}$.
But this contradicts Theorem 2.13: $\Phi(x, 0_W) \cap \left\{-\Phi^*\left(0_{\mathcal{C}(X,Y)}, \Lambda\right) - C_Y \setminus \{0_Y\}\right\} = \emptyset$.
The assumption that $(\bar{\Lambda}, \bar{y})$ is not a maximizer of $(D\ 1)$ (a $\Lambda \in \mathcal{C}(W,Y)$ is required so that a $y \in -\Phi^*\left(0_{\mathcal{C}(X,Y)}, \Lambda\right)$ exists with $\bar{y} <_Y y$) again allows the analogous construction of a contradiction in relation to Theorem 2.13. □

The *minimal element mapping* $H : W \rightrightarrows Y$ is defined as

$$H(w) = \mathrm{Min}\left(\bigcup_{x \in X} \Phi(x, w), \leq_Y\right) = \mathrm{Min}\left(\Phi(X, w), \leq_Y\right) \text{ if } H(w) \neq \emptyset.$$

For all $w \in W$ the set $H(w)$ is exactly the set of minimal elements of the image set for problem $(P\ 3)$ and

$$H(0_W) = \mathrm{Min}\left(\Phi(X, 0_W), \leq_Y\right) = \mathrm{Min}\left(F(X), \leq_Y\right)$$

Is the set of minimal elements of the image set for problem $(P\ 2)$. $H(0_W)$ is a direct result of the definition of Φ.

For an arbitrary $w \in W$ $H(w)$ is *externally stable* if

$$\operatorname{Min}\left(\bigcup_{x\in X}\Phi(x,w),\leq_Y\right) \text{ is externally stable in the sense of Definition 2.8.}$$

The following Lemma 2.16 refers to Lemma 6.1.3 in Sawaragi, Nakayama and Tanino [22]:

Lemma 2.16

For all $\Lambda \in \mathcal{C}(W,Y)$ with $\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)$, $H^*(\Lambda) \neq \emptyset$:

$\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right) \subset H^*(\Lambda)$. If $H(w)$ is externally stable for all $w \in W$, then $\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right) = H^*(\Lambda)$.

Proof:

Let $\Lambda \in \mathcal{C}(X,Y)$ with $\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)$, $H^*(\Lambda) \neq \emptyset$ be arbitrarily chosen. Then (separation formulation):

$$\begin{aligned}\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right) &= \operatorname{Max}\left(\bigcup_{x\in X,\ w\in W}\{\Lambda(w)\}-\Phi(x,w),\leq_Y\right)\\ &= \operatorname{Max}\left(\bigcup_{w\in W}\left(\{\Lambda(w)\}-\bigcup_{x\in X}\Phi(x,w)\right),\leq_Y\right)\end{aligned}$$

and

$$\begin{aligned}H^*(\Lambda) &= \operatorname{Max}\left(\bigcup_{w\in W}\{\Lambda(w)\}-H(w),\leq_Y\right)\\ &= \operatorname{Max}\left(\bigcup_{w\in W}\{\Lambda(w)\}-\operatorname{Min}\left(\bigcup_{x\in X}\Phi(x,w),\leq_Y\right),\leq_Y\right)\\ &= \operatorname{Max}\left(\bigcup_{w\in W}\{\Lambda(w)\}+\operatorname{Max}\left(-\bigcup_{x\in X}\Phi(x,w),\leq_Y\right),\leq_Y\right).\end{aligned}$$

Applying Lemma 2.10 for $F: W \rightrightarrows Y$, $F(w)=\{\Lambda(w)\}$ and $G: W \rightrightarrows Y$, $G(w) = -\bigcup_{x\in X}\Phi(x,w)$, this gives the result

$$\left.\begin{aligned}&\operatorname{Max}\left(\bigcup_{w\in W}\{\Lambda(w)\}-\bigcup_{x\in X}\Phi(x,w),\leq_Y\right)\\&\subset\operatorname{Max}\left(\bigcup_{w\in W}\{\Lambda(w)\}+\operatorname{Max}\left(-\bigcup_{x\in X}\Phi(x,w),\leq_Y\right),\leq_Y\right).\end{aligned}\right\}\quad(2.3)$$

Therefore $\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)\subset H^*(\Lambda)$. If $\operatorname{Min}\left(\bigcup_{x\in X}\Phi(x,w),\leq_Y\right)$ is externally stable, i.e. $\operatorname{Min}\left(\bigcup_{x\in X}\Phi(x,w),\leq_Y\right)\subset\operatorname{Min}\left(\operatorname{Min}\left(\bigcup_{x\in X}\Phi(x,w),\leq_Y\right),\leq_Y\right)+C_Y$, then this is also true for $\operatorname{Max}\left(-\bigcup_{x\in X}\Phi(x,w),\leq_Y\right)$ for all $w\in W$, i.e. $\operatorname{Max}\left(-\bigcup_{x\in X}\Phi(x,w),\leq_Y\right)\subset\operatorname{Max}\left(\operatorname{Max}\left(-\bigcup_{x\in X}\Phi(x,w),\leq_Y\right),\leq_Y\right)-C_Y$. In this case (2.3) is satisfied, so that $\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)=H^*(\Lambda)$. □

In the following it is assumed that $H(w)\neq\varnothing$ is externally stable for all $w\in W$. It then follows that $H^*(\Lambda)=\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right)\neq\varnothing$ and the dual problem $(D\ 1)$ can be equivalently written:

$$\leq_Y-\max_{\substack{\text{with the constraints}\\ \Lambda\in\mathcal{C}(W,Y)}}\bigcup -H^*(\Lambda).$$

Lemma 2.17

Assuming that $H(w)\neq\varnothing$ is externally stable for all $w\in W$, then for $H^{**}(0_W)\neq\varnothing$:

$$\operatorname{Max}\left(\bigcup_{\Lambda\in\mathcal{C}(W,Y)}-\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right),\leq_Y\right)=H^{**}(0_W).$$

Proof:

It then follows from Lemma 2.16 that

$$H^{**}(0_W)=\operatorname{Max}\left(\bigcup_{\Lambda\in\mathcal{C}(W,Y)}-H^*(\Lambda),\leq_Y\right)=\operatorname{Max}\left(\bigcup_{\Lambda\in\mathcal{C}(W,Y)}-\Phi^*\left(0_{\mathcal{C}(X,Y)},\Lambda\right),\leq_Y\right),$$

because $H(w)\neq\varnothing$ is externally stable for all $w\in W$,
giving the desired equality. □

Since $H(0_W)$ and $H^{**}(0_W)$ are the sets of the minimal and maximal elements for the primal and dual problems, the duality properties can be represented by the relations between $H(0_W)$ and $H^{**}(0_W)$. Strong duality applies for $(P\ 2)$ to $(D\ 1)$ if $H(0_W) = H^{**}(0_W)$, and weaker versions result for $H(0_W) \subset H^{**}(0_W)$.

The following Definition 2.18 refers to Definition 6.1.3 in Sawaragi, Nakayama and Tanino [22]:

Definition 2.18
Problem $(P\ 2)$ is *stable with respect to the perturbation mapping* Φ if the minimal value map H is subdifferentiable at the point 0_W.

Definition 2.18 points out the analogy to the stability definition of vector-valued optimality problems. This definition (see Sawaragi, Nakayama and Tanino [22] in finite dimensional spaces and vector-valued target functions) is substantiated in Proposition 2.6: H is subdifferentiable at the point 0_W exactly when $H(0_W) \subset H^{**}(0_W)$. Stability equivalently characterizes a strong duality and is formulated in the following theorem with the concept of the minimizer.

Theorem 2.19 extends Theorem 6.1.1 of Sawaragi, Nakayama and Tanino [22].

Theorem 2.19 (strong duality for (*P* 2) and (*D* 1))
Let us assume the external stability of $H(w)$ for all $w \in W$. Problem $(P\ 2)$ is then stable exactly when for every element $\bar{x} \in X$ of $(P\ 2)$ and every element $\bar{y} \in F(\bar{x})$ with $(\bar{x}, \bar{y})$ as minimizer of $(P\ 2)$ an element $\bar{\Lambda} \in \mathcal{C}(W,Y)$ of $(D\ 1)$ exists so that $\bar{y} \in -\Phi^{*}\left(0_{\mathcal{C}(X,Y)}, \bar{\Lambda}\right)$ and $(\bar{\Lambda}, \bar{y})$ is also a maximizer of $(D\ 1)$.

Proof:
"$\Rightarrow$": Assume that $(P\ 2)$ is stable. Then H is subdifferentiable in 0_W and according to Proposition 2.6 this is equivalent to $H(0_W) \subset H^{**}(0_W)$. Then choose an arbitrary $\bar{x} \in X$ and a $\bar{y} \in F(\bar{x})$ so that $\bar{y} \in H(0_W)$. The existence of $\bar{\Lambda} \in \mathcal{C}(W,Y)$ then follows from the external stability of Lemma 2.17

$$\left(\operatorname{Max}\left(\bigcup_{\Lambda \in \mathcal{C}(W,Y)} -\Phi^{*}\left(0_{\mathcal{C}(X,Y)}, \Lambda\right), \leq_Y \right) = H^{**}(0_W) \text{ with } H^{**}(0_W) \neq \varnothing \right)$$

and $H(0_W) \subset H^{**}(0_W)$, so that $\bar{y} \in -\Phi^{*}\left(0_{\mathcal{C}(X,Y)}, \bar{\Lambda}\right)$. Furthermore, it follows from Corollary 2.15 that $(\bar{\Lambda}, \bar{y})$ is a maximizer of $(D\ 5)$.

"$\Leftarrow$": The conclusions above can be applied in inverse order (beginning with $\overline{\Lambda} \in \mathcal{C}(W,Y)$ and $(\overline{\Lambda}, \overline{y})$ a maximizer of $(D\ 1)$ with Lemma 2.17) in order to arrive at $H(0_W) \subset H^{**}(0_W)$. This implies that H is subdifferentiable at 0_W, proving the stability of $(P\ 2)$. □

Theorem 2.19 shows a strong duality assertion, since here the existence of an ordinary element $\overline{y}$ in the target values of the primal and dual set-valued problems is utilized.

In the proof of Theorem 2.19 the external stability of $H(w)$ is assumed for all $w \in W$. Strong duality is therefore achieved under strong assumptions.

Note:
In Neukel [18] further optimality concepts are applied in a socio-economic context.

3. Socio-economic interpretation of the conjugate duality – compensation payments for aircraft noise in urban conflict situations

Urban building planning in the area surrounding an airport must ensure the orderly development of urban construction. The existing historically based and permanent conflict situations originating in the development of the settlement structure raise the question of the extent to which the existing structures for fundamental urban building principles require "overplanning" for changes to building law in consideration of increased aircraft noise. Individual measures (a compensation payment for property owners in the aircraft noise zones) can be seen as a solution if an improvement is achieved with a comprehensive procedure for dealing with the urban conflict situation. See the concept of a minimizer and conjugate duality in Section 2.

Urban building development measures focus increasingly on the development of urban districts or other areas of the urban region or also on a new development within the scope of urban building restructuring. These measures are frequently applied when required in the interest of the general public, for example a district solution improvement in the region surrounding an airport. Closely spaced residential construction, conflict situations between residential areas and the airport or lacking developments and infrastructures require urban building interventions in certain regions. Here, the set-valued optimization of Section 2 offers far-reaching and (partial) order-related potentials for the improvement of such situations.

Conjugate duality offers the potential to optimally represent the relationships between urban building functions and uses in a conflict situation. The duality criteria to which the optimality is oriented can differ. The problem consists of considering the chosen optimality criteria and order relations that in the ideal case complement each other in urban building planning. With the help of the duality approach for urban building planning, for example in the region surrounding the Frankfurt airport, the conflicts arising can be minimized. Complex conflict situations require a division into different "zones" with selective dual properties and order relations. The functional structure of cities is subject to continuous change. Restrictions are affected by permanent changes for urban structural optimization, so that conjugate duality with a partial order offers a possible solution approach. An exact separation of urban building, sociological, economic and housing policy aspects is not possible in a conflict situation described by a duality model. All parameters enter into the calculation. The application of the minimizer in the duality model is also useful (see Neukel [17] and [19].

In this section the conjugate duality model of Section 2 is causatively investigated with a specially chosen order relation for the solution of a conflict situation in the region surrounding the airport in Frankfurt am Main.

The following model utilizes the Lagrange duality, however together with the concept of perturbation mapping.

According to OP-Online [11] the Federal State of Hesse intends to pay a total of 22.5 million Euros in compensation to communities subjected to aircraft noise. For example, Neu-Isenburg profits from these compensation payments with 285,000 Euros and Offenbach with 393,000 Euros. These are the communities most seriously affected by aircraft noise and having only a few advantages from airport business.

The expressions below derive from Section 2, with $X, W, Y := \mathbb{R}$. A hypothetical non-cooperative situation is examined, i.e. two interest groups and their strategy sets *X* and *W*, with a payment function $\Phi : X \times W \rightrightarrows Y$ $\Phi(x,w) := \{x, w \in [0,10] | -28500x + 285000 + w \leq 285000\}$ (perturbation map of the conjugate duality for Neu-Isenburg). *Y* is a cost framework, and the profit of a group (positive payment) implies a loss in the other group (negative payment). This arbitrarily chosen perturbation map is linear and follows an economy-based demand function in relation to Neu-Isenburg for $w = 0_W$. Group 1 (the airport operator) chooses a strategy $x \in X$ and Group 2 (the residents in the region surrounding the airport or the town of Neu-Isenburg) selects $w \in W$. Here the operator wants to minimize Φ in relation to *x*, while on the other hand the residents want to maximize Φ in relation to *w*.

The operator must solve the problem

$$\leq_Y -\min_{x \in X} \Phi(x,w)$$

$$= \leq_Y -\min_{x \in X} \{x, w \in [0,10] | -28500x + 285000 + w \leq 285000\}$$

(see also Problem $(P\ 3)$) for $w = 0_W$ and the residents must solve the problem

$$\leq_Y -\max_{w \in W} \Phi(x,w)$$

$$= \leq_Y -\max_{w \in W} \{x, w \in [0,10] | -28500x + 285000 + w \leq 285000\}.$$

Maple 17 yields the results below for both problems (the residents in the region surrounding the airport or the town of Neu-Isenburg will attempt to influence the airport operator in order to optimize $x \in X$):

$minimize(-28500x + 285000, x = 0..10, location)$

$$0, \{[\{x = 10\}, 0]\}$$

$maximize(-28500x + 285000 + w, w = 0, location)$

$$-28500x + 285000, \{[\{\ \}, -28500x + 285000]\}$$

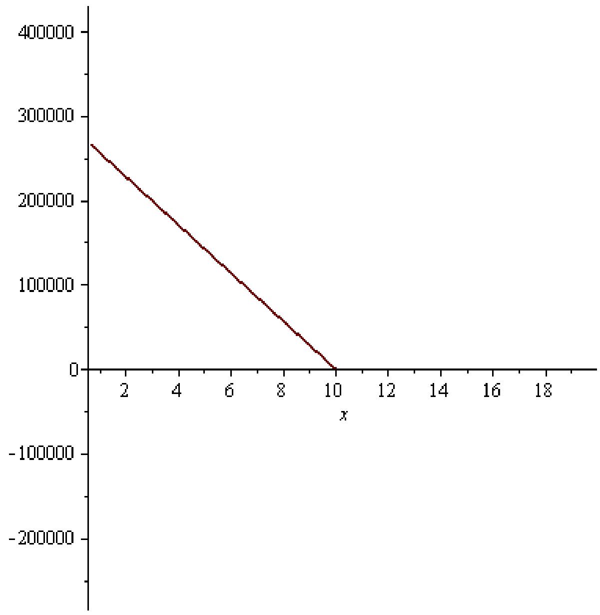
$maximize(-28500x + 285000, x = 0..10, location)$

$$285000, \{[\{x = 0\}, 285000]\}$$

$seq(-28500x + 285000, x = 0..10)$

285000, 256500, 228000, 199500, 171000, 142500, 114000, 85500, 57000, 28500, 0

$plot(-28500x + 285000, x = 0..10)$



Initially it is assumed that the operator makes the first decision. The residents in the region surrounding the airport then respond based on the knowledge of the operator's decision. The residents in the region surrounding the airport want to maximize payment and therefore choose $\Lambda \in \mathcal{C}(W,Y)$ so that the payment is exactly

$$\Phi^*(T,\Lambda)$$

$$= \operatorname{Max}\left(\bigcup_{x\in X,\, w\in W} \{T(x) - \Lambda(w)\} - \Phi(x,w), \leq_Y \right)$$

$$:= \operatorname{Max}\left(\bigcup_{x\in X,\, w\in W} \{x, w \in [0,10] | -39300x + 393000 - \Lambda(w) \leq 393000\} - \Phi(x,w), \leq_Y \right)$$

with $T \in \mathcal{C}(X,Y)$ and $\Lambda \in \mathcal{C}(W,Y)$ (this requires that the maximal set is non-empty). But $\operatorname{Max}\left(\bigcup_{x\in X,\, w\in W} \{T(x) - \Lambda(w)\} - \Phi(x,w), \leq_Y \right)$ also depends on $x \in X$, namely on the airport operator's decision. In turn, the operator chooses $x \in X$ so as to limit its damage and disadvantage to the lowest amount possible, resulting in the "dual" payment

$$\leq_Y - \max_{\substack{\text{with the constraints} \\ \Lambda \in \mathcal{C}(W,Y)}} - \Phi^* \left(0_{\mathcal{C}(X,Y)}, \Lambda\right)$$

$$= \leq_Y - \max_{\substack{\text{with the constraints} \\ \Lambda \in \mathcal{C}(W,Y)}} - \operatorname{Max}\left(\bigcup_{x \in X,\, w \in W} \left\{ x \in [0.10] \middle| -\Lambda(0) \leq 393000 \right\} - \Phi(x, 0_w), \leq_Y \right)$$

$$= \leq_Y - \max_{\substack{\text{with the constraints} \\ \Lambda \in \mathcal{C}(W,Y)}} - \operatorname{Max}\left(\bigcup_{x \in X,\, w \in W} \left\{ x \in [0.10] \middle| -\Lambda(0_w) \leq 393000 \right\} \right.$$

$$\left. - \left\{ x \in [0.10] \middle| -28500x + 285000 \leq 285000 \right\}, \leq_Y \right)$$

(see Problem $(D\ 1)$ for the special case $w = 0_W$. ($(P\ 3)$ then agrees $(P\ 2)$). The case $w > 0_W$ is not examined further here.

For the maximization problem above Maple 17 yields:

$M := maximize(-\Lambda - (-28500\,x + 285000), x = 0\,..10, location)$

$-\Lambda, \{[\{x = 10\}, -\Lambda]\}$

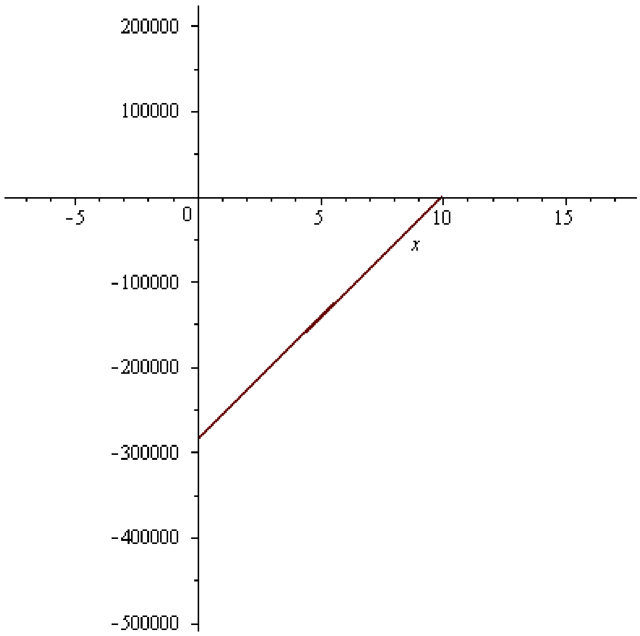
$-M$

$\Lambda, -\{[\{x = 10\}, -\Lambda]\}$

$seq(28500\,x - 285000, x = 0\,..10)$

$-285000, -256500, -228000, -199500, -171000, -142500, -114000, -85500, -57000, -28500, 0$

$plot(28500\,x - 285000, x = 0\,..10)$



For a group it is therefore advantageous to choose as the second group with the early knowledge of the opposite group's initial decision. The "perturbation" is initiated by the second group. In the special case $w = 0_W$ the duality statements of Theorem 2.13 and Theorem 2.19 apply. For Neu-Isenburg Λ is then the negotiation variable, also with reference to the neighboring city of Offenbach.

4. Resumé and outlook

This book utilizes a mathematical set-valued perturbation approach of duality for the first time in relation to social transformation processes relating to future development (sustainability). The result is a theoretical research-oriented approach for the analysis and evaluation of sustainability-oriented social change processes with regard to compensation payments for aircraft noise in an urban context. The conjugate duality with a specially chosen order relation, a partial order, is a scientifically founded method for an intervention approach for development, planning and implementation in urban building conflict situations. The book shows that the airport noise compensation payments are "set-valued" variable and that no fixed value – contrary to the procedure of the city of Frankfurt am Main [10], according to which a single lump sum is paid for airport noise related disturbances on the basis of a building's market value.

This book investigates minimal and maximal elements for a partial order. This leads to a set-valued duality model, which serves as an example of a decision strategy for the determination of compensation payments for aircraft noise in the Frankfurt am Main region. The application of this duality yields a variable optimal point set.

In this book the currently established practice of public administrations in Germany for aircraft noise compensation payments is extended to include the concept of conjugate duality. The evaluations of these prices in specified regional zones (conflict situations with socio-economic influences) are re-examined in order to extract a dual result. This confirms the optimal strategy.

With reference to the existing urban districts and the development of these aircraft noise compensation payments are elicited and a documented optimum found with the help of a duality (selective monitoring). This approach is groundbreaking for the market participation valuation of the market participants e.g. (process space theory, controllable approach).

References

[1] R.U. Ansari, X.Q. Yang, J.C. Yao, Existence and Duality of Implicit Vector Variational Problems, *Numer. Funct. Anal. Opt.*, **22** (2001), no. 7-8, 815 - 829. https://doi.org/10.1081/NFA-100108310

[2] A.Y. Azimov, R.N. Gasimov, On Weak Conjugacy, Weak Subdifferentials and Duality with Zero Gap in Nonconvex Optimization, *Int. J. Appl. Math.*, **1** (1999), no. 2, 171 - 192.

[3] A.Y. Azimov, R.N. Gasimov, Stability and Duality of Nonconvex Problems via Augemted Lagrangian, *Cybernet Systems Anal.*, **38** (2002), 412 - 421. https://doi.org/10.1023/A:1020316811823

[4] R.I. Bot, S.-M. Grad, G. Wanka, *Duality in Vector Optimization*, Springer-Verlag, Berlin/Heidelberg, 2009. https://doi.org/10.1007/978-3-642-02886-1

[5] S. Brumelle, Duality for Multiple Objective Convex Programs, *Math. Methods Operations Res.*, **6** (1981), no. 2, 159 - 172. https://doi.org/10.1287/moor.6.2.159

[6] W. Fenchel, On Conjugate Convex Functions, *Com. J. Math.*, **1** (1949), 73 - 77. https://doi.org/10.4153/cjm-1949-007-x

[7] A. Hamel, A Duality Theory for Set-Valued Functions I: Fenchel Conjugation Theory, *Set-Val. Anal. Var.*, **17** (2009), no. 2, 153 - 182. https://doi.org/10.1007/s11228-009-0109-0

[8] A. Hamel, C. Schrage, Notes about Extended Real- and Set-Valued Functions, *Journ. Conv. Anal.*, **19** (2012), no. 2, 355 - 384.

[9] E. Hernández, L. Rodríguez-Marín, Nonconvex Scalarization in Set Optimization with Set-Valued Maps, *J. Math. Anal. Appl.*, **325** (2007), no. 1, 1 - 18. https://doi.org/10.1016/j.jmaa.2006.01.033

[10] https://www.frankfurt.de/sixcms/detail.php?id=3060&_ffmpar[_id_inhalt]=11840712, aufgerufen am 20.12.2017

[11] https://www.op-online.de/hessen/hessen-will-225-millionen-euro-kommunen-zahlen-unter-fluglaerm-leiden-8706818.html, aufgerufen am 1.12.2017

[12] R. Kasimbeyli, M. Mammadov, On Weak Subdifferntials, Directional Derivatives and Radial Epiderivatives for Nonconvex Functions, *SIAM J. Optim.*, **20** (2009), no. 2, 841 - 855. https://doi.org/10.1137/080738106

[13] H. Kawasaki, A Duality Theorem in Multiobjective Nonlinear Programming, *Math. Operations Res.*, **7** (1982), no. 1, 95 - 110. https://doi.org/10.1287/moor.7.1.95

[14] Y. Kücük, I. Atasever, M. Kücük, Generalized Weak Subdifferentials, *Optimization*, **60** (2011), no. 5, 337 - 552. https://doi.org/10.1080/02331930903524670

[15] C.S. Lalitha, R. Arora, Conjugate Maps, Subgradients and Conjugate Duality in Set-Valued Optimization, *Num. Funct. Analysis and Optimization*, **28** (2007), no. 7-8, 897 – 909. https://doi.org/10.1080/01630560701501073

[16] D.T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, Vol. 319, Springer-Verlag, Berlin/Heidelberg/New York, 1989. https://doi.org/10.1007/978-3-642-50280-4

[17] N. Neukel, A New Approach for the Determination of Publicly Registered Land Values on the Basis of Interval-valued Duality Theory and Regression, *Appl. Math. Sciences*, **11** (2017), no. 16, 783 - 805. https://doi.org/10.12988/ams.2017.7128

[18] N. Neukel, Dualitätskonzepte und Sozio-ökonomische Paradigmen mit Ordnungsrelationen zur Mengenoptimierung, Dissertation, FAU Erlangen-Nuremberg, 2016.

[19] N. Neukel, Extension of a Dual Equivalence Class Model and its Application to the Socio-Economy of the German Real Estate Market, *Applied Mathematical Sciences*, **11** (2017), no. 47, 2325 – 2340. https://doi.org/10.12988/ams.2017.78254

[20] N. Neukel, Order Relations of Sets and its Application in Socio-Economics, *Appl. Math. Sciences*, **7** (2013), no. 115, 5711 - 5 739. https://doi.org/10.12988/ams.2013.37419

[21] R.T. Rockafellar, Conjugate Duality and Optimization, Chapter in *CBMS-NSF Regional Conference Series Applied Mathematics*, SIAM, Philadelphia, 1974. https://doi.org/10.1137/1.9781611970524.ch1

[22] Y. Sawaragi, H. Nakayama, T. Tanino, *Theory of Multiobjective Optimization*, Vol. 176, Mathematics in Science and Engineering, Academic Press Inc., London, UK, 1st ed, 1985.

[23] R. Schiel, Vector Optimization and Control with Partial Differential Equations and Pointwise State Constraints, Dissertation, FAU Erlangen-Nuremberg, 2014.

[24] C. Schrage, *Set-Valued Analysis*, Diss., Martin-Luther-Universität Halle-Wittenberg, 2009.

[25] W. Song, Duality for Vector Optimization of Set-Valued Functions, *J. Math. Anal. Appl.*, **201** (1996), no. 1, 212 - 225. https://doi.org/10.1006/jmaa.1996.0251

[26] T. Tanino, Conjugate Duality in Vector Optimization, *J. Math. Anal. Appl.*, **167** (1992), no. 1, 84 - 97. https://doi.org/10.1016/0022-247X(92)90237-8

[27] T. Tanino, On Supremum of a Set in Multi-Dimensional Spaces, *J. Math. Anal. Appl.*, **130** (1988), no. 2, 386 - 397. https://doi.org/10.1016/0022-247X(88)90314-9

[28] T. Tanino, Y. Sawaragi, Conjugate Maps and Duality in Multiobjective Optimization, *J. Optim. Theory Appl.*, **31** (1980), no. 4, 473 – 499. https://doi.org/10.1007/BF00934473

[29] W. Vogel, *Vektoroptimierung in Produkträumen*, Mathematical Systems in Economics No. 35, Verlag Anton Hain, Meisenheim am Glan, 1977.